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CUM EIUS USU

IN ANALYSI FINITORUM

AC

DOCTRINA SERIERUM

AUCTORE

LEONARDO EULERO

ACAD. REG. SCIENT. ET ELEG. LITT. BORUSS. DIRECTORE
PROP. HONOR. ACAD. IMP. SCIENT. PETROP.
ET ACADEMIARUM REGIARUM PARISINAE
ET LONDINENSIS SOCIO.

TICINI
IN TYPOGRAPHEO PETRI GALEATII
Superiorum permiſſu.
1787.

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EULERIANUM . OPUS . A . SE . EDITUM

ET . IUVENTUTIS . BONO . PARATUM

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EDITOR LECTORI.

*A*BSOLUTISSIMUM *Differentialis* Calculi *Opus*, *quod trigintaduobus abhinc annis Geometrarum princeps LEONARDUS EULERUS Berolinensibus typis, & Petropolitanae Academiae ſumptibus vulgaverat, distractis iam cunctis ferme exemplaribus, nec nisi ingenti pretio comparandis, novam iampridem editionem postulabat. Sed ingratum hunc molestumque laborem ut ipse susciperem, a me nunquam impetrare potuissem, nisi votis & flagitationibus illorum, qui diutius hoc opere carere non poterant, accessissent doctorum Virorum invitationes, cohortationesque amicorum, qui rem Geometris gratam, Iuventuti utilissimam augurantes, me diu haesitantem & ancipitem stimulis additis pepulerunt.*

Ad rem itaque aggressus Eulerianum Opus a capite ad calcem quam potui diligentissime legi & perlustravi, calculosque omnes analyticos, quodque operosius erat ingentes illos numerorum acervos, qui passim occurrunt, maxima qua potui circumspectione recognovi. Mirum dictu est, quot quantosque offenderim

typo-

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typographicos errores corruptos numeros vitiatos calculos ambiguos sensus, ut ferre aequo animo haud poſſim Berolinensis Typographi simulationem dicam an levitatem? qui bina tantum errata, eaque levissima emendanda proposuit: Μέμνησο ἀπιστεῖν. *Correctis typothetarum mendis reliquum erat, ut de pretio ac praestantia huic editioni prae antecedenti comparanda cogitarem, eamque novis quibusdam additamentis locupletarem. Igitur elegans, & grave Auctoris Elogium ab eximio Geometra & Philoſopho Marchione CONDORCETO exaratum, & in publico Academiae Scientiarum Pariſiensis conventu recitatum, sed nondum in eius Academiae Actis publici iuris factum, Operi univerſo praepoſui: tum in operis calce indicem adieci abſolutissimum omnium EULERI Lucubrationum, cum earum, quae ſeparatim editae iuſtis voluminibus continentur, tum earum, quae numero prope incredibili in plurium Academiarum Commentariis leguntur, tum denique caeterarum, quae ineditae ad bis centum usque inter eius manuscripta inventae sunt, quaeque Petropolitanae Academiae traditae in eius Commentariis deinceps in lucem emittentur. Tam amplum atque immensum mathematicae eruditionis thesaurum prae oculis habere illumque pro opportunitate consulere res erit procul dubio iunioribus provectioribusque Geometris utilissima, qui ex infini-*

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finita illa rerum tractatarum multitudine quid sibi sumant, quid omittant statim perspicient, & ab iis quaestionibus tractandis abstinebunt, quas cognoverint ab **EULERO** *inchoatas simul & perfectas, neque tanquam novum proponent quod ipse occupaverit; quod caeteroquin evenire facile posset, quum nullum ferme sit in universo Disciplinarum Mathematicarum ambitu argumentum paullo elegantius, quod Geometra ipſe Briaraeus vel non delibaverit, vel non absolverit.*

Postremo operae pretium duxi brevibus quibusdam annotationibus locum unum vel alterum Euleriani Operis illuſtrare, eaque perpauca huc derivare, quae poſtmodum ab Auctore ipso, vel a Geometris aliis ad Differentialis Calculi amplificationem inventa sunt; perpauca inquam, nam Euleriano scripto multa addere hominis esset vel incredibilem huius Geometrae foecunditatem ignorantis, vel Phidiae simulacrum, aut Apellis tabulam emendare sibi arrogantis

Superesset nunc, ut grata animi significatione declararem quid ipse debeam Viro celeberrimo & polyhistori Praeceptori nuper meo GREGORIO FONTANAE in Regio Ticinensi Archigymnaſio ſublimioris Matheseos Professori, quo suasore & auspice rem omnem molitus sum; sed verecundiae hominis modestissimi, quem vel nominare omnino prohibeor, parcendum eſt.

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ÉLOGE
DE M. EULER

Prononcé à la rentrée de l'Académie royale des ſciences le 6 Février 1785

Par

M. LE MARQ. DE CONDORCET

Sécretaire perpétuel de l'Académie des ſciences, de l'Inſtitut de Bologne des Académies de Petersbourg, de Turin &c. &c. &c.

LÉONARD EULER, Directeur de la claſſe de mathématique dans l'Académie de Petersbourg, & auparavant dans celle de Berlin, de la Société royale de Londres, des Académies de Turin, de Lisbonne & de Basle, aſſocié étranger de celle des Sciences, naquit à Basle le 15 Avril 1707, de PAUL EULER & de MARGUERITTE BRUCKER.

Son pere, devenu en 1708 paſteur du village de Riechen près de Basle, fut ſon premier inſtituteur, & eut bientôt le plaiſir de voir les eſpérances des talens & de la gloire d'un fils, ſi douces pour un cœur paternel, naître & ſe fortifier ſous ſes yeux & par ſes ſoins.

Il avoit étudié les mathématiques ſous JAQUES BERNOULLI. On ſait que cet homme illuſtre joignoit à un grand génie

b pour

pour les ſciences une philoſophie profonde, qui n' accompagne pas toujours ce génie, mais qui ſert à lui donner plus d' étendue & à le rendre plus utile. Dans ſes leçons, il faiſoit ſentir à ſes diſciples que la géométrie n' eſt pas une ſcience iſolée, & la leur préſentoit comme la baſe & la clef de toutes les connoiſſances humaines; comme la ſcience où l' on peut le mieux obſerver la marche de l' eſprit; celle dont la culture exerce le plus utilement nos facultés, puiſqu' elle donne à l' entendement de la force & de la juſteſſe à la fois; enfin comme une étude également précieuſe par le nombre ou la variété de ſes applications & par l' avantage de faire contracter l' habitude d' une méthode de raiſonner, qui peut s' employer enſuite à la recherche des vérités de tous les genres, & nous guider dans la conduite de la vie.

PAUL EULER, pénétré des principes de ſon maître, enſeigna les élémens des mathématiques à ſon fils, quoiqu' il le deſtinât à l' étude de la théologie; & lorsque le jeune EULER fut envoyé à l' univerſité de Basle, il ſe trouva digne de recevoir les leçons de JEAN BERNOULLI. Sa facilité, ſon application, ſes diſpoſitions heureuſes, lui mériterent bientôt l' amitié de DANIEL & de NICOLAS BERNOULLI, diſciples & déja rivaux de leur pere. Il eut même le bonheur d' obtenir celle du ſévere JEAN BERNOULLI, qui voulut bien lui donner, une fois par ſemaine, une leçon particuliere, deſtinée à éclaircir les difficultés qui ſe préſentoient à lui

dans

dans le cours de ſes lectures & de ſes travaux. Les autres jours etoient employés par M. EULER, à ſe metre en état de profiter de cette faveur ſignalée.

Cette méthode excellente empêchoit ſon génie naiſſant de s'épuiſer contre des obſtacles invincibles, de s'égarer dans les routes nouvelles qu'il cherchoit à s'ouvrir; elle guidoit & ſecondoit ſes efforts: mais en même tems elle l'obligeoit de déployer toutes ſes forces, qu'il augmentoit encore par un exercice proportionné à ſon âge & à l'étendue de ſes connoiſſances.

Il ne jouit pas longtems de cet avantage; & à peine eut-il obtenu le titre de maître-és-arts, que ſon perre, qui le deſtinoit à lui ſuccéder, l'obligea de quitter les mathématiques, pour la théologie. Heureuſement, cette rigueur ne fut que paſſagere. On lui fit aiſément entendere que ſon fils étoit né pour remplacer dans l'Europe JEAN BERNOULLI, & non pour être paſteur de Riechen.

Un ouvrage que M. EULER fit à 19 ans ſur la mâture des vaiſſeaux, ſujet propoſé par l'Académie des Sciences, obtint un acceſſit en 1727; honneur d'autant plus grand, que le jeune habitant des Alpes n'avoit pu être aidé par aucune connoiſſance pratique, & qu'il n'avoit été vaincu que par M. BOUGUER, Géomêtre habile, alors dans la force de ſon talent, & déja depuis dix ans Profeſſeur d'hydrographie dans une ville maritime.

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M. EULER concourroit en même tems pour une chaire dans l'Univerſité de Basle; mais c'eſt le ſort qui prononce entre les ſavans admis à diſputer ces places, & il ne fut pas favorable, nous ne diſons point à M. EULER, mais à ſa patrie, qui le perdit peu de jours aprés & pour toujours. Deux ans auparavant, NICOLAS & DANIEL BERNOULLI avoient été appellés en Ruſſie. M. EULER, qui les vit partir avec regret, obtint d'eux la promeſſe de chercher à lui procurer le même honneur qu'il ambitionnoit de partager, & il ne faut pas en être ſurpris. La ſplendeur de la capitale d'un grand empire, cet éclat qui, ſe rependant ſur les travaux dont elle eſt le théatre & ſur les hommes qui l'habitent, ſemble ajouter à leur gloire, peuvent aiſément ſeduire la jeuneſſe & frapper le citoyen libre, mais obſcur & pauvre d'une petite république. MM. BERNOULLI furent fideles a leur parole & ſe donnerent, pour avoir auprés d'eux un concourrent ſi redoutable, autant de ſoins que des hommes ordinaires en auroient pu prendre pour écarter leurs rivaux.

Le voyage de M. EULER fut entrepris ſous de triſtes auſpices. Il apprit bientôt que NICOLAS BERNOULLI avoit déja été victime de la rigueur du climat. Et le jour même où il entra ſur les terres de l'empire Ruſſe fut celui de la mort de CATHERINE I., événement qui parût d'abord menacer d'une diſſolution prochaine l'Académie, dont cette Princeſſe, fidele aux vues de ſon Epoux, venoit d'achever

la

la fondation. M. EULER, éloigné de ſa patrie, n'ayant point, comme M. DANIEL BERNOULLI, à y rapporter un nom célebre & reſpecté, prit la réſolution d'entrer dans la marine ruſſe. Un des Amiraux de PIERRE I. lui avoit déja promis une place, lorſque, heureuſement pour la géométrie, l'orage élevé contre les ſciences ſe diſſipa. M. EULER obtint le titre de Profeſſeur, ſuccéda en 1733 à M. DANIEL BERNOULLI, lorſque cet homme illuſtre ſe retira dans ſon pays; & la même année il épouſa M.[lle] GSELL, ſa compatriote, fille d'un peintre que PIERRE I. avoit ramené en Ruſſie, au retour de ſon premier voyage. Dés-lors pour nous ſervir de l'expreſſion de BACON, M. EULER ſentit qu'il avoit donné des otages à la fortune, & que le pays, où il pouvoit eſpérer de former un établiſſement pour ſa famille, étoit devenu pour lui une patrie néceſſaire. Né chez une Nation où tous les gouvernemens conſervent au moins l'apparence & le langage des conſtitutions républicaines, où, malgré des diſtinctions plus réelles que celles qui ſéparent les premiers eſclaves d'un deſpote, du dernier de ſes ſujets, on a ſoigneuſement gardé toutes les formes de l'égalité, où le reſpect qu'on doit aux loix s'étend juſqu'aux uſages les plus indifférens, pourvu que l'antiquité, où l'opinion populaire, les ait conſacrés; M. EULER ſe trouvoit tranſporté dans un pays où le Prince exerce une autorité ſans bornes, où la loi la plus ſacrée des gouvernemens abſolus, celle qui règle la ſucceſſion à l'empire, étoit

alors

alors incertaine & mépriſée, où des chefs, eſclaves du ſouverain, regnoient deſpotiquement ſur un peuple eſclave; & c'étoit dans le moment où cet empire, gouverné par un étranger ambitieux, défiant & cruel, gémiſſoit ſous la tyrannie de BIREN, & offroit un ſpectacle auſſi effrayant qu'inſtructif aux ſavans, qui étoient venus chercher dans ſon ſein la gloire, la fortune & la liberté de goûter en paix les douceurs de l'étude.

On ſent tout ce que dut éprouver l'ame de M. EULER, lié à ce ſéjour par une chaine qu'il ne pouvoit plus rompre. Peut-etre doit on à cette circonſtance de ſa vie, cette opiniâtreté pour le travail, dont il prit alors l'habitude & qui devint ſon unique reſſource dans une capitale où l'on ne trovoit plus que des ennemis, ou des ſatellites du miniſtre, occupés de flatter ſes ſoupçons, ou de s'y dérober. Cett impreſſion fut ſi forte ſur M. EULER, qu'il la conſervoit encore, lorsqu'en 1741, l'année d'après la chûte de BIREN, dont la tyrannie fit place à un gouvernement plus modéré & plus humain, il quitta Petersbourg pour ſe rendre a Berlin, où le Roi de Pruſſe l'avoit appellé. Il fut préſenté à la Reine-mere. Cette Princeſſe ſe plaiſoit dans la converſation des hommes éclairés, & elle les accueilloit avec cette familiarité noble, qui annonce dans les princes le ſentiment d'une grandeur perſonelle, indépendante de leurs titres, & qui eſt devenue un des caracteres de cette Famille auguſte. Cependant

dant la Reine de Pruſſe ne put obtenir de M. EULER que des monoſyllabes. Elle lui reprocha cette timidité, cet embarras, qu'elle croyoit ne pas mériter d'inſpirer. Pourquoi ne voulez-vous donc pas me parler, lui dit-elle? Madame, répondit-il, parceque je viens d'un pays où, quand on parle, on eſt pendu.

Parvenu au moment de rendre compte des travaux immenſes de M. EULER, j'ai ſenti l'impoſſibilité d'en ſuivre les détails, de faire connoitre cette foule de découvertes, de méthodes nouvelles, de vues ingénieuſes répandues dans plus de trente ourages publiés à part, & dans prés de ſept-cens mémoires, dont environ deux-cens dépoſés à l'Académie de Petersbourg avant ſa mort, ſont deſtinés à enrichir ſucceſſivement la collection qu'elle publie.

Mais un caractere particulier m'a ſemblé le diſtinguer des hommes illuſtres, qui, en ſuivant la même carriere, ont obtenu une gloire que la ſienne n'a pas éclipſée : c'eſt d'avoir embraſſé les ſciences mathématiques dans leur univerſalité, d'en avoir ſucceſſivement perfectionné les différentes parties, &, en les enrichiſſant toutes par des découvertes importantes, d'avoir produit une révolution utile dans la maniere de les traiter. J'ai donc cru qu'en formant un tableau méthodique des différentes branches de ces ſciences, en marquant pour chacune les progrès, les changemens heureux, qu'elle doit au génie de M. EULER, j'aurois, du moins autant

tant que mes forces me le permettent, donné une idée plus juſte de cet homme célebre, qui, par la reunion de tant de qualités extraordinaires, a été, pour ainſi dire, un phénomêne, dont l'hiſtoire des ſciences ne nous avoit encore offert aucun exemple.

L'Algébre n'avoit été pendant longtems qu'une ſcience très-bornée. Cette maniere de ne conſidérer l'idée de la grandeur que dans le dernier dégré d'abſtraction, où l'eſprit humain puiſſe atteindre, la rigueur avec la quelle on ſépare de cette idée tout ce qui, en occupant l'imagination, pourroit donner quelqu'appui, ou quelque repos à l'intelligence; enfin l'extrême généralité des ſignes que cette ſcience emploit, la rendent en quelque ſorte étrangere à notre nature, trop éloignée de nos conceptions communes, pour que l'eſprit humain pût aiſément s'y plaire & en acquérir facilement l'habitude. La marche même des méthodes algébriques rebutoit encore les hommes les plus propres à ces meditations. Pour peu que l'objet qu'on pourſuit ſoit compliqué, elles forcent de l'oublier totalement, pour ne ſonger qu'à leurs formules; la route qu'on ſuit eſt aſſurée, mais le but où l'on veut arriver, le point d'où l'on eſt parti, diſparoiſſent également aux regards du Géomêtre; & il a fallu longtems du courage pour oſer perdre la terre de vue & s'expoſer ſur la foi d'une ſcience nouvelle. Auſſi, en jettant les yeux ſur les ouvrages des grands Géomêtres du ſiecle dernier, de ceux mê-

même, aux quels l'algebre doit les découvertes les plus importantes, on verra combien peu ils étoient accoûtumés à manier ce même inſtrument qu'ils ont tant perfectionné, & l'on ne pourra s'empêcher de regarder comme l'ouvrage de M. EULER, la révolution, qui a rendu l'analyſe algébrique une méthode lumineuſe, univerſelle, applicable à tout & même facile.

Aprés avoir donné ſur la forme des racines des équations algébriques, ſur leur ſolution générale, ſur l'élimination, pluſieurs théories nouvelles & des vues ingénieuſes ou profondes, M. EULER porta ſes recherches ſur le calcul des quantités tranſcendantes. LEIBNITZ & les deux BERNOULLI ſe partagent la gloire d'avoir introduit dans l'analyſe algébrique les fonctions exponentielles & logarithmiques. COTES avait donné le moyen de repréſenter par des ſinus ou des coſinus les racines de certaines équations algébriques.

Un uſage heureux de ces découvertes conduiſit M. EULER à obſerver les rapports ſinguliers des quantités exponentielles & logarithmiques avec les tranſcendantes nées dans le cercle & enſuite à trouver des méthodes au moyen des quelles faiſant diſparoître de la ſolution des problémes les termes imaginaires qui s'y feroient préſentés & qui auroient embaraſſé le calcul, quoiqu'on ſût qu'ils duſſent ſe détruire, réduiſant les formules à une expreſſion plus ſimple & plus commode, il eſt parvenu à donner une forme entierement nouvelle à

la partie de l' analyſe, qui s'applique aux queſtions d'aſtronomie & de phyſique. Cette forme a été adoptée par tous les Géomêtres, elle eſt devenue d'un uſage commun, & elle a produit dans cette partie du calcul à peu prés la même révolution, que la découverte des logarithmes avoit produite dans les calculs ordinaires.

Ainſi, à certaines époques, où, après de grands efforts, les ſciences mathématiques ſemblent avoir épuiſé toutes les reſſources de l'eſprit humain & atteindre le terme marqué à leurs progrès, tout-à-coup une nouvelle méthode de calcul vient s'introduire dans ces ſciences & leur donner une face nouvelle. Bientôt on les voit s'enrichir rapidement par la ſolution d'un grand nombre de problêmes importans dont les Géomêtres n'avoint oſé s'occuper, rebutés par la difficulté, & pour ainſi dire, par l'impoſſibilité phyſique de conduire leurs calculs jusqu'à un reſultat réel. Peut-être la juſtice exigeroit-elle de réſerver à celui qui a ſçu introduire ces méthodes & les rendre uſuelles, une portion de la gloire de tous ceux qui les employent avec ſuccès; mais du moins il a ſur leur reconnoiſſance des droits, qu'ils ne pourroient conteſter ſans ingratitude.

L'analyſe des ſéries a occupé M. EULER dans preſque toutes les époques de ſa vie. C'eſt même une des parties de ſes ouvrages, où l'on voit briller le plus cette fineſſe, cette ſagacité, cette variété de moyens & de reſſources, qui le caractériſent.

Les

Les fractions continues, inventées par le Vicomte de BROUNKER, paroiſſoient perſqu'oubliées des Géomêtres. M. EULER en perfectiona la théorie, en multiplia les applications & en fit ſentir toute l'importance.

Les recherches, perſqu'abſolument neuves, ſur les ſéries de produits indéfinis, offrent des reſſources néceſſaires à la ſolution d'un grand nombre de queſtions utiles ou curieuſes; & c'eſt ſur-tout en imaginant ainſi de nouvelles formes de ſéries & en les employant non ſeulement à des approximations, dont on eſt ſi ſouvent forcé de ſe contenter, mais auſſi à la découverte des vérités abſolues & rigoureuſes, que M. EULER a ſçu aggrandir cette branche de l'analyſe, aujourd'hui ſi vaſte, & bornée avant lui à un petit nombre de méthodes & d'applications.

Le calcul intégral, l'inſtrument le plus fecond de découvertes que jamais les hommes ayent poſſédé, à changè de face depuis les ouvrages de M. EULER. Il a perfectionné, étendu, ſimplifié toutes les méthodes employées ou propoſées avant lui. On lui doit la ſolution générale des équations linéaires, premier fondement de ces formules d'approximations ſi variées & ſi utiles. Une foulle de méthodes particulieres, fondées ſur differens principes, ſont répandues dans ſes ouvrages & réunies dans ſon traité du calcul integral. Là, on le voit, par un heureux uſage des ſubſtitutions, ou rappeller à une méthode connue des équations qui ſembloient

ſ'y refuſer, ou réduire aux premieres différentielles des équations d'ordres ſupérieurs; tantôt, en conſiderant la forme des intégrales, il en deduit les conditions des équations différentielles, aux quelles elles peuvent ſatisfaire; & tantôt l' examen de la forme des facteurs, qui rendent une différentielle complette, le conduit à former des claſſes générales d'équations intégrables. Quelquefois une propriété particuliere, qu'il remarque dans une équation, lui offre un moyen de ſéparer les indéterminées, qui ſembloient devoir y reſter confondues. Ailleurs, ſi une équation où elles ſont ſéparées ſe dérobe aux méthodes communes, c'eſt en mêlant ces indéterminées qu'il parvient á connoître l'intégrale.

Au premier coup d'œil, le choix & la réuſſite de ces moyens peuvent ſembler en quelque ſorte appartenir au hazard. Cependant un ſuccés ſi fréquent & ſi ſur, oblige de reconnoître une autre cauſe; & il n'eſt pas toujours impoſſible de ſuivre le fil délié, qui a guidé le génie. Si par exemple, on conſidére la forme des ſubſtitutions employées par M. EULER, on découvrira ſouvent ce qui a pu lui faire prévoir que cette opération produiroit l'effet dont il avoit beſoin; & ſi on examine la forme que dans une de ſes plus belles méthodes il ſuppoſe aux facteurs d'une équation du ſecond ordre, on verra qu'il s'eſt arrêté á une de celles qui appartiennent particulierement à cet ordre d'équation. A la verité cette ſuite d'idées, qui dirige alors un analyſte, eſt

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moins une méthode, dont il puiſſe développer la marche, qu'une ſorte d'inſtinct particulier, dont il ſeroit difficile de rendre compte; & ſouvent il aime mieux ne pas faire l'hiſtoire de ſes penſées, que de s'expoſer au ſoupçon d'en avoir donné un roman ingénieux & fait aprés-coup.

M. EULER a obſervé que les équations différentielles ſont ſuſceptibles de ſolutions particulieres qui ne ſont pas compriſes dans la ſolution générale. M. CLAIRAUT a fait la mêmé remarque; mais M. EULER a montré depuis, pourquoi ces intégrales particulieres étoient exclues de la ſolution générale, il eſt le premier qui ſe ſoit occupé de cette théorie, perfectionnée depuis par pluſieurs Géomêtres célebres & dans la quelle le mémoire de M. DE LA GRANGE, ſur la nature de ces intégrales & leur uſage dans la ſolution des problêmes, n'a plus rien laiſſé à deſirer.

Nous citerons encore une partie de ce calcul, qui appartient preſqu'en entier à M. EULER. C'eſt celle où l'on cherche des intégrales particulieres pour une certaine valeur déterminée des inconnues, que renferme l'équation. Cette théorie eſt d'autant plus importante, que ſouvent l'integrale générale fuit abſolument nos recherches, & que dans les problêmes où une valeur approchée de l'intégrale, ne ſuffit pas aux vues qu'on ſe propoſe, la connoiſſance de ces intégrales particulieres peut ſupléer à ce défaut. En effet, on connoit alors, du moins pour certains points, la valeur rigou-

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goureuſe; & cette connoiſſance, unie à celle d'une valeur générale approchée, doit ſuffire à preſque tous les beſoins de l'analyſe.

Perſonne n'a fait un uſage plus étendu & plus heureux des méthodes qui donnent la valeur de plus en plus approchée d'une quantité déterminée, par des équations differentielles & dont on a déja une premiere valeur; & il s'eſt également occupé de donner un moyen direct de deduire immédiatement de l'équation même une valeur aſſés voiſine de la vraie, pour que les puiſſances élevées de leur difference puiſſent être négligées; moyen ſans le quel les methodes d'approximation, en uſage parmi les Géomêtres, ne pourroient s'étendre aux équations pour les quelles les obſervations ou des conſidérations particulieres ne donnent pas cette premiere valeur, dont ces méthodes ſuppoſent la connoiſſance.

Ce que nous avons dit, ſuffit pour montrer jusqu'à quel point M. EULER avoit approfondi la nature des équations différentielles, la ſource des difficultés qui s'oppoſent a l'intégration, & la maniere de les éluder ou de les vaincre. Son grand ouvrage ſur cet objet eſt non ſeulement un recueil précieux de méthodes neuves & étendues; c'eſt encore une mine féconde de decouvertes, que tout homme né avec quelque talent, ne peut parcourir ſans en rapporter de riches dépouilles; & l'on peut dire de cette partie des travaux de M. EULER, comme de beaucoup d'autres, que les mé-

méthodes qu'elle renferme ſerviront longtems après lui à réſoudre des queſtions importantes & dificiles & que ſes ouvrages produiront encore & plus d'une découverte & plus d'une réputation.

Le calcul aux différences finies n'étoit preſque connû que par l'ouvrage obſcur, mais plein de ſagacité, de TAYLOR. M. EULER en fit une branche importante du calcul intégral, lui donna une notation ſimple & commode, & ſçut l'appliquer avec ſuccès à la théorie des ſuites, à la recherche de leurs ſommes ou de l'expreſſion de leurs termes généraux, à celle de la racine des équations déterminées, à la maniere d'avoir, par un calcul facile, la valeur approchée des produits, ou des ſommes indéfinies de certains nombres.

C'eſt à M. D'ALEMBERT qu'appartient réellement la découverte du calcul aux différences partielles, puiſque c'eſt à lui qu'eſt due la connoiſſance des fractions arbitraires, qui entrent dans les intégrales. Mais dans les premiers ouvrages de M. D'ALEMBERT, on voyoit plus le réſultat du calcul, que le calcul lui-même. C'eſt à M. EULER que l'on en doit la notation. Il a ſçu ſe le rendre propre en qualque maniere par la profonde théorie qui l'a conduit a reſoudre un grand nombre de ces équations; à diſtinguer les formes des intégrales, pour les différens ordres & pour les différens nombres de variables; à réduire ces équations, lorſqu'elles ont certaines formes, à des intégrations ordinaires; à donner les moyens de rappeller à ces for-

formes, par d'heureuſes ſubſtitutions, celles qui s'en éloignent; en un mot, en découvrant dans la nature des équations aux différences partielles pluſieurs de ces propriétés ſingulieres, qui en rendent la théorie générale ſi difficile & ſi piquante, qualités preſqu'inſéparables en géométrie, où le dégré de la difficulté eſt ſi ſouvent la meſure de l'intérêt qu'on prend á une queſtion, de l'honneur qu'on attache à une découverte. L'influence d'une vérité nouvelle ſur la ſcience même, ou ſur quelqu'application importante, eſt le ſeul avantage qui puiſſe balancer ce mérite de la difficulté vaincue chez des hommes, pour qui le plaiſir d'appercevoir une vérité eſt toujours proportionné aux efforts qu'elle leur a coutés.

M. EULER. n'avoit négligé aucune partie de l'analyſe. Il a démontré quelques uns des Théorèmes de Fermat, ſur l'analyſe indéterminée, & en a trouvé pluſieurs autres non moins curiuex, non moins difficiles à découvrir. La marche du Cavalier au jeu d'Echecs, & différens autres problêmes de ſituation, ont auſſi piqué ſa curioſité & exercé ſon génie. Il mêloit aux recherches les plus importantes, ces amuſemens ſouvent plus difficiles, mais preſque inutiles & au progrés même de la ſcience & aux applications tentées juſqu'ici. M. EULER avoit un eſprit trop ſage, pour ne pas ſentir l'inconvénient de ſe livrer longtems à ces recherches purement curieuſes; mais trop étendu en même tems pour ne pas voir que leur inutilité ne devoit être que momentanée, & que le

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ſeul moyen de la faire ceſſer étoit de chercher à les approfondir & à les généraliſer.

L'Application de l'Algébre à la Géométrie avoit occupé, depuis DESCARTES, preſque tous les Géomêtres du dernier ſiécle. Mais M. EULER à prouvé qu'ils n'avoient pas, à beaucoup près, tout épuiſé. On lui doit de nouvelles recherches ſur le nombre des points qui déterminent une ligne courbe, dont le dégré eſt connu, & ſur celui des interſections des lignes de différens dégrés. On lui doit également l'équation générale des courbes, dont les développées, les ſecondes, les troiſiémes développées, en un mot les développées d'un ordre quelconque, ſont ſemblables à la courbe génératrice; équation remarquable par ſon extrême ſimplicité.

La Théorie générale des ſurfaces courbes étoit peu connue; & M. EULER eſt le premier qui l'ait développée dans un ouvrage élémentaire. Il y ajouta celle des rayons oſculateurs de ces ſurfaces; & il parvint à cette concluſion ſinguliere, que la courbure d'un élément de ſurface eſt déterminée par deux des rayons-oſculateurs des courbes formées par l'interſection de la ſurface & d'un plan qui paſſe par la perpendiculaire au poin donné; que ces rayons ſont le plus grand & le plus petit de tous ceux qui appartiennent à la ſuite des courbes ainſi formées, & qu'enfin ils ſe trouvent toujours dans des plans perpendiculaires l'un à l'autre.

Il donna de plus une méthode de déterminer les ſurfa-

ces qui peuvent être développées ſur un plan & une théorie des projections géographiques de la ſphere. Ces deux ouvrages renferment une application de calcul des différences partielles à des problêmes géométriques; application qui peut s'étendre à beaucoup de queſtions intéreſſantes, & dont la primiere idée eſt duë à M. EULER.

Ses recherches ſur les courbes, qui, tracées ſur une ſphére, ſont rectifiables algébriquement, & ſur les ſurfaces courbes, dont les parties correſpondantes à des parties d'un plan donné ſont égales entr'elles, l'ont conduit à une nouvelle eſpéce d'analyſe, à la quelle il donne le nom d'analyſe infinitéſimale indeterminée; parceque, comme dans l'analyſe indéterminée ordinaire, les quantités qui reſtent arbitraires ſont aſſujetties à certaines conditions; &, de même que l'analyſe indéterminée a pu ſervir quelque fois á la perfection de l'Algébre, M. EULER regardoit ſa nouvelle analyſe comme une ſcience qui devoit un jour être utile aux progrés du calcul integral.

En effet, ces queſtions iſolées, qui ne tiennent pas au corps méthodique des ſciences mathématiques, qui n'entrent point dans les applications qu'on peut en faire, ne doivent pas être regardées ſeulement comme des moyens d'exercer les forces, ou de faire briller le génie des Géomêtres. Preſque toujours, dans les ſciences, on commence par cultiver ſéparément quelques parties iſolées. A meſure que les décou-

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vertes ſucceſſives ſe multiplient, les liaiſons qui uniſſent ces parties ſe laiſſent ſucceſſivement appercevoir ; & le plus ſouvent c'eſt aux lumieres qui reſultent de cette réunion que ſont dues les grandes découvertes, qui ſont époque dans l'hiſtoire de l'eſprit humain.

La queſtion de déterminer les courbes ou les ſurfaces, pour les quelles certaines fonctions indéfinies ſont plus grandes ou plus petites que pour toutes les autres, avoit exercé les Géomêtres les plus illuſtres du ſiecle dernier ; les ſolutions des problêmes du ſolide de la moindre réſiſtance, de la courbe de plus vîte deſcente, de la plus grande des aires iſopérimetres, avoient été célebres en Europe. La méthode générale de réſoudre le problême, étoit cachée dans ces ſolutions, & ſurtout dans celle que JACQUES BERNOULLI avoit trouvée par la queſtion des iſopérimetres & qui lui avoit donné ſur ſon Frere un avantage, que tant de chefs-d'œuvres, enfantés depuis par JEAN BERNOULLI, n'ont pu faire oublier. Mais il falloit développer cette méthode, il falloit la réduire en formules générales ; & c'eſt ce que fit M. EULER dans un ouvrage imprimé en 1744 & l'un des plus beaux monumens de ſon génie. Pour trouver ces formules, il avoit été obligé d'employer la conſideration des lignes courbes. Quinze ans après, un jeune Géométre (M. de la GRANGE), qui, dans ſes premiers eſſais, annonçoit un digne ſucceſſeur D'EULER, reſolut le même problême par une mé-

thode purement analytique. M. EULER admira le premier ce nouvel effort de l'art du calcul, s'occupa lui-même d'expoſer la nouvelle méthode, d'en préſenter les principes, & d'en donner les développemens avec cette clarté; cette élégance qui brillent dans tous ſes ouvrages. Jamais le génie ne reçut & ne rendit un plus bel hommage; & jamais il ne ſe montra plus ſupérieur à ces petites paſſions, que le partage d'un peu de glorie rend ſi actives & ſi violentes dans les hommes ordinaires.

Nous terminerons cet expoſé des travaux de M. EULER ſur l'analyſe pure, en obſervant qu'il ſeroit injuſte de borner ſon influence ſur les progrés des mathématiques, aux decouvertes ſans nombre dont ſes ouvrages ſont remplis. Ces communications qu'il a ouvertes entre toutes les parties d'une ſcience ſi vaſte; ces vuës générales que ſouvent même il n'indique pas, mais qui n'échappent point á un eſprit attentif; ces routes dont il s'eſt contenté d'ouvrir l'entrée & d'aplanir les premiers obſtacles, ſont encore autant de bienfaits dont les ſciences s'enrichiront, & dont la poſtérité jouira, en oubliant peut-être la main dont elle les aura reçu.

Le traité de Mécanique, que M. EULER donna en 1736, eſt le premier grand ouvrage où l'analyſe ait été appliquée á la ſcience du mouvement. Le nombre des choſes neuves, ou préſentées d'une maniere nouvelle, qui entrent dans ce trai-

traité, eût étonné, les Géomêtres, ſi M. EULER n'en eût déja publié ſéparément la plus grande partie.

Dans ſes nombreux travaux ſur la même ſcience, il fut toujours fidele á l'analyſe & l'uſage heureux qu'il en a fait, a mérité à cette méthode la préférence qu'elle a enfin obtenue ſur toutes les autres.

La ſolution du problême où l'on cherche le mouvement d'un corps lancé dans l'eſpace & attiré vers deux points fixés, eſt devenue célebre par l'art avec le quel des ſubſtitutions dont M. EULER ſavoit ſi bien prévoir la forme, l'ont conduit à réduire aux quadratures des équations, que leur complication & leur forme pouvoieut faire regarder comme inſolubles.

Il appliqua l'analyſe au mouvement d'un corps ſolide, d'une figure donnée; & elle le conduiſit à ce beau Théorême, deja donné par SEGNER, qu'un corps d'une figure quelconque peut tourner librement d'un mouvement uniforme autour de trois axes perpendiculaires entr'eux, á la connoiſſance de pluſieurs propriétés ſingulieres de ces trois axes principaux, & enfin aux équations générales du mouvement d'un corps, quelque ſoit ſa figure & la loi des forces accélératrices, qui agiſſent ſur ſes élémens, ou ſur quelques unes de ſes parties.

Le problême des cordes vibrantes, & tous ceux qui appartiennent à la Théorie du ſon, ou des loix des oſcillations

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de l'air, ont été ſoumis à l'analyſe par les nouvelles méthodes dont il enrichit le calcul des différences partielles, une théorie du mouvement des fluides, appuyée ſus ce même calcul, étonna par la clarté qu'il a repandue ſur des queſtions ſi épineuſes & la facilité qu'il a ſu donner à des méthodes fondées ſur l'analyſe ſi profonde.

Tous les problêmes de l'aſtronomie phyſique, qui ont été traités dans ce ſiecle, ont été réſolus par des méthodes analytiques particulieres à M. EULER. Sa Théorie de la lune eſt un modele de la ſimplicité, de la préciſion aux quelles on peut porter ces méthodes; &, en liſant cet ouvrage, on n'eſt pas moins étonné de voir juſqu'où un homme d'un grand génie, animé du déſir de ne rien laiſſer à faire ſur une queſtion importante, peut pouſſer la patience & l'opiniâtreté du travail.

L'aſtronomie n'employoit que des methodes géométriques, M. EULER ſentit tout ce qu'elle pouvoit eſpérer des ſecours de l'analyſe; & il le prouva par des exemples, qui, imités depuis par pluſieurs ſavans célebres, pourront un jour faire prendre à cette ſcience une forme nouvelle.

Il embraſſa la ſcience navale dans un grand ouvrage, au quel une ſavante analyſe ſert de baſe, & où les queſtions les plus difficiles ſont ſoumiſes à ces methodes générales & ſecondes qu'il ſavoit ſi bien créer & employer. Longtems aprés, il publia ſur la même matiere, un abrégé élémentaire de

de ce même traité, où il renferme ſous la forme la plus ſimple ce qui peut être utile á la pratique & ce que doivent ſavoir ceux qui ſe conſacrent au ſervice de mer. Cet ouvrage, quoique deſtiné par l'auteur aux ſeules écoles de l'empire de Ruſſie, lui merita une gratification du Roi, qui jugea que des travaux utiles à tous les hommes, avoient des droits à la reconnoiſſance de tous les ſouverains, & voulût montrer que même aux extrémités de l'Europe, des talens ſi rares ne pouvoient échapper ni à ſes regards, ni à ſes bienfaits M. EULER fut ſenſible à cette marque de l'eſtime d'un Roi puiſſant & elle reçut un nouveau prix à ſes yeux, de la main qui la lui transmit; c'étoit celle de M. TURGOT, Miniſtre reſpecté dans l'Europe par ſes lumieres, comme par ſes vertus, fait pour commander à l'opinion, plutôt que pour lui obéir, & dont le ſuffrage, toujours dicté par la vérité & jamais par le déſir d'attirer ſur lui même l'approbation publique, pouvoit flatter un ſage trop accoutumé à la gloire, pour être encore ſenſible au bruit de ſa renommée.

Dans les hommes d'un génie ſuperieur, l'extrême ſimplicité de caractere peut s'allier avec les qualités de l'eſprit, qui ſemblent le plus annoncer de l'habileté, ou de la fineſſe.

Auſſi M. EULER, malgré cette ſimplicité qui ne ſe démentit jamais, ſavoit cependant diſtinguer, avec une ſagacité toujours indulgente, il eſt vrai, les hommages d'une admiration

tion éclairée & ceux que la vanité prodigue aux grands hommes, pour s'aſſurer du moins le mérite de l'enthouſiaſme.

Les travaux ſur la Dioptrique, ſont fondés ſur une analyſe moins profonde; & on eſt tenté de lui en ſavoir gré, comme d'une eſpece de ſacrifice. Les différens rayons, dont un rayon ſolaire eſt formé, ſubiſſent dans le méme milieu des refractions différentes; ſéparés ainſi des rayons voiſins, ils paroiſſent ſeuls, ou moins mélangés, & donnent la ſenſation de couleur qui leur eſt propre. Cette refrangibilité varie dans les différens milieux pour chaque rayon & ſuivant une loi, qui n'eſt pas la même que celle de la refraction moyenne dans ces milieux. Cette obſervation donnoit lieu de croire que deux priſmes inégaux & de différentes matieres, combinés enſemble, pourroient détourner un rayon de ſa route ſans le décompoſer, ou plutôt en replaçant par une triple refraction les rayons élémentaires dans une direction parallele.

De la vérité de cette conjecture pouvoit dépendre, dans les lunettes, la deſtruction des iris qui colorent les objets vus à travers les verres lenticulaires. M. EULER étoit convaincu de la poſſibilité du ſuccés, d'aprés cette idée méthaphiſique, que ſi l'oeil a été compoſé de diverſes humeurs, c'eſt uniquement dans l'intention d'y détruire les effets de l'aberration de refrangibilité. Il ne s'agiſſoit donc que de chercher à imiter l'opération de la nature; & il en propoſa

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les moyens, d'après une théorie qu'il s'étoit formée. Ses premiers eſſais exciterent les Phyſiciens à s'occuper d'un objet qu'ils paroiſſoient avoir negligé. Leurs expériences ne confirmerent point la théorie de M. EULER : mais elles confirmerent les vues qu'il avoit eues ſur la perfection des lunettes. Inſtruit alors par eux des loix de la diſperſion dans les différens milieux, il abandonna ſes premieres idées, ſoumit au calcul les réſultats de leurs expériences & enrichit la Dioptrique de formules analytiques ſimples, commodes, générales, applicables à tous les inſtrumens qu'on peut conſtruire.

On a encore de M. EULER quelques eſſais ſur la théorie générale de la lumiere, dont il cherchoit à concilier les phénomenes avec les lois des oſcillations d'une fluide, parceque l'hypothéſe de l'émiſſion des rayons en ligne droite, lui paroiſſoit préſenter des difficultés inſurmontables. La Théorie de l'aiman, celle de la cohéſion des corps, celles des frottemens, furent auſſi pour lui l'occaſion de ſavans calculs, appuyés malheureuſement ſur des hypothéſes, plutôt que ſur des expériences.

Le calcul des probabilités, l'arithmétique politique, furent encore l'objet da ſes infatigables travaux. Nous ne citerons ici que ſes recherches ſur les tables de mortalité & ſur les moyens de les déduire des phénomenes avec plus d'exactitude. Sa méthode de prendre un milieu entre des obſervations, ſes calculs ſur l'établiſſement d'une caiſſe d'em-

prunt, dont le but eſt d'aſſurer aux veuves, aux enfans, ou une ſomme fixe, ou une rente, payables aprés la mort d'un mari, ou d'un pere: moyen ingénieux, imaginé par des Géométres-Philoſophes, pour contrebalancer le mal moral, qui réſulte de l'établiſſement des rentes viageres, & pour rendre utiles aux familles les plus petites épargnes que leur chef peut faire ſur ſon gain journalier, ou ſur les appointemens ſoit d'une commiſſion, ſoit d'une place.

On a vu dans l'éloge de M. DANIEL BERNOULLI, qu'il avoit partagé avec M. EULER ſeul la gloire d'avoir remporté dix prix à l'académie des ſciences. Souvent ils travaillerent pour les mêmes ſujets, & l'honneur de l'emporter ſur ſon concurrent fut encore partagé entr'eux, ſans que jamais cette rivalité ait ſuſpendu les temoignages reciproques de leur eſtime, ou refroidi le ſentiment de leur amitié. En examinant les ſujets ſur les quels l'un ou l'autre ont obtenu la victoire, on voit que le ſuccés a dépendu ſurtout du caractere de leur talent; lorſque la queſtion exigeoit de l'adreſſe dans la maniere de l'enviſager, un uſage heureux de l'expérience, ou des vues de phyſique ingénieuſes & neuves l'avantage étoit pour M. DANIEL BERNOULLI; n'offroit elle a vaincre que de grandes difficultés de calcul, falloit-il créer de nouvelles méthodes d'analyſe, c'étoit M. EULER qui l'emportoit; & ſi l'on pouvoit avoir la témérité de vouloir juger entr'eux, ce ne ſeroit pas entre deux hommes qu'on

qu'on auroit à prononcer, ce ſeroit entre deux genres d'eſprit, entre deux manieres d'employer le génie.

Nous n'aurions donné qu'une idée trés-imparfaite de la fécondité de M. EULER, ſi nous n'ajoutions à cette foible eſquiſſe de ſes travaux, qu'il eſt peu de ſujets importans pour les quels il ne ſoit revenu ſur ſes traces, en refaiſant même pluſieurs fois ſon premier ouvrage. Tantôt il ſubſtituoit une méthode directe & analytique à une méthode indirecte. Tantôt il étendoit ſa premiere ſolution à des cas qui lui avoient d'abord échappé; ajoutant preſque toujours de nouveaux exemples, qu'il ſavoit choiſir avec un art ſingulier, parmi ceux qui offroient ou quelque application utile, ou quelque remarque curieuſe. La ſeule intention de donner à ſon travail une forme plus méthodique, d'y repandre plus de clarté, d'y ajouter un nouveau degré de ſimplicité, ſuffiſoit pour le déterminer à des travaux immenſes. Jamais Géométre n'a tant écrit; & jamais aucun n'a donné à ſes ouvrages un tel degré de perfection. Lorsqu'il publioit un mémoire ſur un objet nouveau, il expoſoit avec ſimplicité la route qu'il avoit parcourue, il en faiſoit obſerver les difficultés, ou les détours; & aprés avoir fait ſuivre ſcrupuleuſement à ſes lecteurs la marche de ſon eſprit dans ſes premiers eſſais, il leur montroit enſuite comment il étoit parvenu à trouver une route plus ſimple. On voit qu'il préféroit l'inſtruction de ſes diſciples, a la petite ſatisfaction de

les étonner, & qu'il croyoit n'en pas faire aſſés pour la ſcience, s'il n'ajoutoit aux vérités nouvelles, dont il l'enrichiſſoit, l'expoſition naïve des idées qui l'y avoient conduit.

Cette méthode d'embraſſer ainſi toutes les branches des mathématiques, d'avoir pour ainſi dire toujours préſentes à l'eſprit toutes les queſtions & toutes les théories, étoit pour M. EULER une ſource de découvertes fermée pour preſque tous les autres, ouverte pour lui ſeul. Ainſi, dans la ſuite de ſes travaux, tantôt s'offroit à lui une méthode ſinguliere d'integrer des équations en les différentiant, tantôt une remarque ſur une queſtion d'analyſe, ou de mécanique, le conduiſoit à la ſolution d'une équation différentielle très-compliquée, qui échappoit aux méthodes directes. C'eſt quelquefois un problême en apparence très difficile, qu'il réſout en un inſtant par une méthode très ſimple, ou un problême qui paroit élementaire & dont la ſolution a des difficultés qu'il ne peut vaincre que par de grands efforts; d'autres fois, des combinaiſons de nombres ſinguliers, des ſéries d'une forme nouvelle, lui préſentent des queſtions piquantes, par leur nouveauté, ou le menent à des vérités inattendues; M. EULER avertiſſoit alors avec ſoin que c'étoit au hazard qu'il devoit les découvertes de ce genre. Ce n'étoit pas en diminuer le mérite; car on voyoit aiſément que ce hazard ne pourroit arriver qu'à un homme qui joindroit à une va-

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ſte étendue de connoiſſances, la ſagacité la plus rare. D'ailleurs, peut-être ne faudroit-il pas le louer de cette candeur, quand même elle lui auroit couté un peu de ſa gloire. Les hommes d'un grand génie ont rarement ces petites ruſes de l'amour propre, qui ne ſervent qu'à rapetiſſer aux yeux des juges èclairés ceux qu'elles agrandiſſent dans l'opinion de la multitude, ſoit que l'homme de génie ſente qu'il ne ſera jamais plus grand qu'en ſe montrant tel qu'il eſt, ſoit que l'opinion n'ait pas ſur lui cet empire qu'elle exerce avec tant de tyrannie ſur les hommes médiocres.

Lorſque l'on lit la vie d'un grand homme, ſoit conviction de l'imperfection attachée à la foibleſſe humaine, ſoit que la juſtice dont nous ſommes capables ne puiſſe atteindre juſqu'à reconnoitre dans nos ſemblables une ſupériorité dont rien ne nous conſole, ſoit enfin que l'idée de la perfection dans un autre, nous bleſſe, ou nous humilie encore plus que celle de la grandeur; il ſemble qu'on a beſoin de trouver un endroit foible; on cherche quelque défaut qui puiſſe nous relever à nos propres yeux, & l'on eſt involontairement porté à ſe défier de la ſincérité de l'Ecrivain, s'il ne nous montre pas cet endroit foible, s'il ne ſouleve point le voile important dont ces défauts ſont couverts.

M. EULER poroiſſoit quelquefois ne s'occuper que du plaiſir de calculer & regarder le point de mécanique ou de phyſique, qu'il examinoit ſeulement comme une occaſion d'exer-

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d'exercer ſon génie & de ſe livrer à ſa paſſion dominante. Auſſi les ſavans lui ont-ils reproché d'avoir quelquefois prodigué ſon calcul à des hypotheſes phyſiques, ou même à des principes métaphyſiques, dont il n'avoit pas aſſés examiné ou la vraiſemblance, ou la ſolidité. Ils lui reprochoient auſſi de s'être trop repoſé ſur les reſſources du calcul & d'avoir négligé celles que pouvoit lui donner l'examen des queſtions même qu'il ſe propoſoit de réſoudre. Nous conviendrons que le premier reproche n'étoit pas ſans fondement. Nous avouerons que dans M. EULER, le Metaphyſicien, ou même le Phyſicien, n'a pas été ſi grand, que le Géométre; & l'on doit regreter ſans doute que pluſieurs parties de ſes ouvrages, par exemple de ceux qu'il a faits ſur la ſcience navale, ſur l'artillerie, n'ayent preſque été utiles qu'aux progrés de la ſcience du calcul. Mais nous croyons que le ſecond reproche eſt beaucoup moins mérité. Partout, dans les ouvrages de M. EULER, on le voit occupé d'ajouter aux richeſſes de l'analyſe, d'en étendre & d'en multiplier les applications. En même tems qu'elle paroit ſon inſtrument unique, on voit qu'il a voulu en faire un inſtrument univerſel. Le progrés naturel des ſciences mathématiques devoit amener cette révolution; mais il l'a vu, pour ainſi dire, s'accomplir ſous ſes yeux; c'eſt à ſon génie que nous la devons; elle a été le prix de ſes efforts & de ſes découvertes. Ainſi, lors même qu'il paroiſſoit abuſer de l'analyſe, & en épuiſer

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tous les ſecrets pour réſoudre une queſtion dont quelques reflexions, étrangeres au calcul, lui euſſent donné une ſolution ſimple & facile, ſouvent il ne cherchoit qu'à montrer les forces & les reſſources de ſon art; & on doit lui pardonner ſi quelquefois, en paroiſſant s'occuper d'une autre ſcience, c'étoit encore au progrés & à la propagation de l'analyſe que ſes travaux étoient conſacrés, puiſque la révolution qui en a été le fruit eſt un de ſes premiers droits à la reconnoiſſance des hommes & un de ſes plus beaux titres à la gloire.

Je n'ai pas cru devoir interrompre le détail des travaux de M. EULER, par le recit des événemens tres-ſimples & trés peu multipliés de ſa vie.

Il s'établit à Berlin en 1741 & y reſta juſqu'en 1766.

Madame la Princeſſe D'ANHALT-DESSAU, Niece du Roi de Pruſſe, voulut recevoir de lui quelques leçons de phyſique. Ces leçons ont été publiées ſous le nom de lettres à une Princeſſe d'Allemagne: ouvrage précieux par la clarté ſinguliere avec laquelle il y a expoſé les vérités les plus importantes de la mécanique, de l'aſtronomie phyſique, de l'optique & de la théorie des ſons; & par des vues ingénieuſes moins philoſophiques, mais plus ſavantes que celles qui ont fait ſurvivre le livre de la pluralité du monde, au ſyſtéme des tourbillons. Le nom d'EULER, ſi grand dans les ſciences, l'idée impoſante que l'on ſe forme de ſes ouvrages

ges deſtinés à approfondir ce que l'analyſe a de plus épineux & de plus abſtrait, donne à ces lettres ſi ſimples, ſi faciles, un charme ſingulier. Ceux qui n'ont pas étudié les mathématiques, étonnés, flattés peut-être de pouvoir entendre un ouvrage d'EULER, lui ſavent gré de s'être mis à leur portée; & ces détails élémentaires des ſciences, acquierent une ſorte de grandeur, par le raprochement qu'on en fait avec la gloire & le génie de l'homme illuſtre, qui les a tracées.

Le Roi de Pruſſe employa M. EULER à des calculs ſur les monnoies, à la conduite des eaux de Sans-ſouci, à l'examen de pluſieurs canaux de navigation. Ce Prince n'étoit pas né pour croire que de grands talens & des connoiſſances profondes, fuſſent jamais des qualités ſuperflues & dangereuſes; & le bonheur de pouvoir être utile, un avantage reſervé par la nature à l'ignorance & à la mediocrité.

En 1750, M. EULER fit le voyage de Francfort pour y recevoir ſa mere, veuve alors, & la ramener à Berlin. Il eut le bonheur de l'y conſerver juſqu'en 1761. Pendant onze ans, elle jouit de la gloire de ſon fils, comme le cœur d'une mere ſait en jouir, & fut plus heureuſe encore peut-être par ſes ſoins tendres & aſſidus dont cette gloire augmentoit le prix.

Ce fut pendant ſon ſéjour à Berlin, que M. EULER, lié par la reconnoiſſance à M. MAUPERTUIS, ſe crut obligé de defendre ce principe de la moindre action, ſur lequel

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le Préſident de l'académie de Pruſſe avoit fondé l'eſpérance d'une ſi grande renommée. Le moyen que choiſit M. EULER ne pouvoit guere être employé que par lui. C'étoit de réſoudre par ce principe quelques uns des problêmes les plus difficiles de la mécanique. Ainſi, dans les tems fabuleux, les Dieux daignoient fabriquer pour les guerriers, qu'ils favoriſoient, des armes impénétrables aux coups de leurs adverſaires. Nous deſirerions que la reconnoiſſance de M. EULER ſe fut bornée à une protection ſi noble & ſi digne de lui; mais on ne peut ſe diſſimuler qu'il n'ait montré trop de dureté dans ſes réponſes à KOENIG: & c'eſt avec douleur que nous ſommes obligés de compter un grand homme parmi les ennemis d'un ſavant malheureux & perſécuté. Heureuſement, toute la vie de M. EULER le met à l'abri d'un ſoupçon plus grave. Sans cette indifférence pour la renommée, qu'il a montrée conſtamment, on auroit pu croire que les plaiſanteries d'un illuſtre partiſan de M. EULER (plaiſanteries que M. DE VOLTAIRE lui-même a depuis condamnées à un juſte oubli) avoient altéré le caractere de M. EULER. Mais s'il fit alors une faute, c'eſt à l'excès ſeul de la reconnoiſſance qu'il faut l'attribuer; & c'eſt par un ſentiment reſpectable, qu'il a été injuſte une ſeule fois dans ſa vie.

Les Ruſſes ayant pénétré dans la Marche en 1760, pillerent une métairie que M. EULER avoit auprés de Charlottembourg; mais le Général TOTTLEBEN n'étoit pas venu

faire la guerre aux ſciences. Inſtruit de la perte que M. EULER avoit eſſuyée, il s'empreſſa de la reparer, en faiſant payer le dommage à un prix fort au deſſus de la valeur réelle; & il rendit compte de ce manque d'égards involontaire à l'Impératrice ELISABETH, qui ajouta un don de quatre mille florins, à une indemnité déja beaucoup plus que ſuffiſante. Ce trait n'a point été connu en Europe; & nous citons avec enthouſiaſme quelques actions ſemblables, que les anciens nous ons tranſmiſes. Cette différence dans nos jugemens, n'eſt-elle pas une preuve de ces progrés heureux de l'eſpece humaine, que quelques Ecrivains s'obſtinent à nier encore apparemment pour éviter qu'on ne les accuſe d'en avoir été les complices?

Le Gouvernement de Ruſſie n'avoit jamais traité M. EULER comme un étranger; une partie de ſes appointemens lui fut toujours payée malgré ſon abſence; & l'Impératrice l'ayant appellé en *1766*, il conſentit à retourner à Petersbourg.

En *1735*, les efforts que lui avoit couté un calcul aſtronomique, pour le quel les autres Académiciens demandoient pluſieurs mois, & qu'il acheva en peu de jours, lui avoient cauſé une maladie, ſuivie de la perte d'un œil. Il avoit lieu de craindre une cécité complette, s'il s'expoſoit de nouveau dans un climat dont l'influence lui êtoit contraire. L'intérét de ſes enfans l'emporta ſur cette crainte; &, ſi on ſonge que l'étude étoit pour M. EULER une paſſion ex-

excluſive, on jugera ſans doute que peu d'exemples d'amour paternel ont mieux prouvé qu'il eſt la plus douce de nos affections.

Il eſſuya, peu d'années après, le malheur qu'il avoit prévû; mais il conſerva, heureuſement pour lui & pour les ſciences, la faculté de diſtinguer encore de grands caracteres tracés ſur une ardoiſe avec de la craie. Ses fils, ſes éleves, copioient ſes calculs, écrivoient ſous ſa dictée le reſte de ſes mémoires; & ſi on en juge par leur nombre & ſouvent par le génie qu'on y retrouve, on pourroit croire que l'abſence encore plus abſolue de toute diſtraction, & la nouvelle énergie que ce recueillement forcé donoit à toutes ſes facultés, lui ont fait plus gagner que l'affoibliſſement de ſa vue n'a pu lui faire perdre de facilité & de moyens pour le travail.

D'ailleurs, M. EULER, par la nature de ſon génie, par l'habitude de ſa vie, s'etoit même involontairement préparé des reſſources extraordinaires, en examinant ces grandes formules analytiques, ſi rares avant lui, ſi frequentes dans ſes ouvrages, dont la combinaiſon & le dévelopement réuniſſent tant de ſimplicité & d'élégance, dont la forme même plait aux yeux come à l'eſprit, on voit qu'elles ne ſont pas le fruit d'un calcul tracé ſur le papier, & que produites toutes entieres dans ſa tête, elles y ont été créées par une imagination également puiſſante & active. Il exiſte dans l'analyſe (& M. EULER en a beaucoup multiplié le nombre)

des formules d'une application commune, & preſque journaliere. Il les avoit toujours préſentes à l'eſprit, les ſavoit par cœur, les recitoit même de mémoire; & M. D'ALEMBERT, lorsqu'il vit M. EULER à Berlin, fut étonné de ce phénomene, nouveau même pour lui. Enfin ſa facilité à calculer de tête étoit portée à un degré qu'on croiroit à peine, ſi l'hiſtoire de ſes travaux n'avoit accoutumé aux prodiges. On l'a vû, dans l'intention d'exercer ſon petit fils aux extractions de racines, ſe former la table des ſix premieres puiſſances de tous les nombres, depuis un juſqu'à cent & la conſerver exactemens dans ſa mémoire. Deux de ſes diſciples avoient calculé, juſqu'au dixſeptieme terme, une ſérie convergente aſſés compliquée. Leurs réſultats, quoique formés d'aprés un calcul écrit, différoient d'une unité au cinquantiéme chiffre. Ils firent part de cette diſpute à leur maitre. M. EULER refit le calcul entier dans ſa tête, & ſa déciſion ſe trouva conforme à la vérité.

Depuis la perte de ſa vue, il n'avoit d'autre amuſement que de faire des aimans artificiels, & de donner des leçons de mathématiques a un de ſes petits-fils, qui lui paroiſſoit annoncer d'heureuſes diſpoſitions.

Il alloit encore quelque fois à l'Académie, principalement dans les circoſtances difficiles, où il croyoit que ſa préſence pouvoit être utile pour y maintenir la liberté. On ſent combien un Préſident perpétuel nommé par la Cour

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peut troubler le repos d'une Académie, & tout ce qu'elle en doit craindre, lorſque, n'étant pas choiſi dans la claſſe des ſavans, il ne ſe ſent pas même arrêté par le beſoin qu'a ſa réputation du ſuffrage de ſes confreres. Et comment des hommes, uniquement occupés de leurs paiſibles travaux & ne ſachant parler que le langage des ſciences, pourroient-ils alors ſe défendre, ſurtout ſi étrangers, iſolés, éloignés de leur patrie, ils tiennent tout du Gouvernement, au quel ils ont à demander juſtice contre le chef que ce Gouvernement même leur a donné?

Mais il eſt un degré de gloire où l'on ſe trouve au deſſus de la crainte: c'eſt lorſque l'Europe entiere s'éleveroit contre une injure perſonnelle faite à un grand-homme, qu'il peut ſans riſque déployer contre l'injuſtice l'autorité de ſa renommée & élever en faveur des ſciences une voix qu'on ne peut empêcher de ſe faire entendre. M. EULER, tout ſimple, tout modeſte qu'il étoit, ſentoit ſes forces, & les a plus d'une fois heureuſement employées.

En 1771, la ville de Petersbourg éprouva un incendie terrible. Les flammes gagnerent la maiſon de M. EULER. Un Baslois, M. PIERRE GRIMON, (dont le nom mérite ſans doute d'être conſervé) apprend le danger de ſon illuſtre compatriote, aveugle & ſouffrant, il ſe précipite au travers des flammes, pénêtre juſqu'à lui, le charge ſur ſes épaules & le ſauve au péril de ſa vie. Sa Bibliothéque, ſes meubles fu-

furent conſumés. Mais les ſoins empreſſés du Comte ORLOFF ſauverent ſes manuſcrits; & cette attention, au milieu du trouble & des horreurs de ce grand déſaſtre, eſt un des hommages les plus vrais & les plus flatteurs, que jamais l'autorité publique ait rendu au génie des ſciences. La maiſon de M. EULER étoit un des bienfaits de l'Impératrice, & un nouveau bienfait en répara promptement la perte. Il a eu de ſa premiere femme treize enfans, dont huit morts en bas-age. Ses trois fils lui ont ſurvécu; & il eût le malheur de perdre ſes deux filles dans la derniere année de ſa vie. De trente huit petits enfans, vingt-ſix vivoient encore à l'époque de ſa mort. En 1776, il épouſa en ſecondes noces, Mlle. GSELL, ſœur de pere de ſa premiere femme. Il avoit gardé toute la ſimplicité de mœurs dont la maiſon paternelle lui avoit donné l'exemple. Tant qu'il a conſervé la vie, il raſſembloit tous les ſoirs, pour la priere commune, ſes enfans, ſes petits enfans, ſes domeſtiques & ceux de ſes éleves qui logeoient chez lui. Il leur liſoit un chapitre de la Bible, & quelque fois accompagnoit cette lecture d'une exhortation.

Il étoit trés religieux; on a de lui une preuve nouvelle de l'exiſtence de Dieu, & de la ſpiritualité de l'ame. Cette derniere même a été adoptée dans pluſieurs écoles de Théologie. Il avoit conſervé ſcrupuleuſement la religion de ſon pays, qui eſt le Calviniſme rigide; & il ne paroit pas qu'à l'exem-

l'exemple de la plupart des Savans proteſtans, il ſe ſoit permis d'adopter des opinions particulieres & de ſe former un ſyſtême de religion.

Son érudition étoit très étendue, ſurtout dans l'hiſtoire des mathématiques. On a prétendu qu'il avoit porté ſa curioſité juſqu'à s'inſtruire des procédés & des regles de l'aſtrologie & que même il en avoit fait quelques applications. Cependant, lorſqu'en 1740 on lui donna ordre de faire l'horoſcope du Prince YVAN, il repréſenta que cette fonction appartenoit à M. KRAAF, qui, en qualité d'aſtronome de la cour, fut obligé de la remplir. Cette crédulité, qu'on eſt étonné de trouver à cette époque dans la cour de Ruſſie, étoit générale, un ſiecle auparavant, dans toutes les Cours de l'Europe. Celles de l'Aſie n'en ont pas encore briſé le joug; & il faut avouer que, ſi on en excepte les maximes communes de la morale, il n'y a juſqu'ici aucune vérité qui puiſſe ſe glorifier d'avoir été adoptée auſſi généralement & auſſi longtems que beaucoup d'erreurs, ou ridicules ou funeſtes.

M. EULER avoit étudié preſque toutes les branches de la phyſique, l'anatomie, la chymie, la botanique; mais ſa ſupériorité dans les mathématiques ne lui permettoit pas d'attacher la plus petite importance à ſes connoiſſances dans les autres genres, quoiqu'aſſez étendues, pour qu'un homme, plus ſuſceptible des petiteſſes de l'amour-propre, eût pû aſpirer à une ſorte d'univerſalité. L'é-

L'étude de la littérature ancienne & des langues ſavantes avoit fait partie de ſon éducation. Il en conſerva le gout toute ſa vie; & n'oublia rien de ce qu'il avoit appris; mais il n'eût jamais ni le tems, ni le déſir d'ajouter à ſes premieres études. Il n'avoit pas lû les poëtes modernes & ſavoit par cœur l'Eneïde.

Mais M. EULER ne perdoit pas de vue les mathèmatiques, même lorſqu'il récitoit les vers de Virgile. Tout étoit propre á lui rappeller cet objet preſque unique de ſes penſées, & on trouve dans ſes ouvrages un ſavant mémoire ſur une queſtion de mécanique, dont il racontoit qu'un vers de l'Eneïde lui avoit donné la premiere idée.

On a dit que, pour les hommes d'un grand talent, le plaiſir du travail en étoit une récompenſe plus douce encore que la gloire. Si cette vérité avoit beſoin d'être prouvée par des exemples, celui de M. EULER. ne permetroit plus d'en douter.

Jamais dans ſes ſavantes diſcuſſions avec de célebres Géomêtres, il n'a laiſſé échapper un ſeul trait qui puiſſe faire ſoupçonner qu'il ſe ſoit occupé des intérêts de ſon amour-propre. Jamais il n'a reclamé aucune de ſes découvertes; &, ſi on revendiquoit quelque choſe dans ſes ouvrages, il s'empreſſoit de réparer une injuſtice involontaire, ſans même trop examiner ſi l'équitè rigoureuſe exigeoit de lui un abandon abſolu. Y avoit-on relevé quelqu'erreur, ſi le reproche étoit

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mal fondé, il l'oublioit; s'il étoit juſte, il ſe corrigeoit, & ne ſongeoit méme pas à obſerver que ſouvent le mérite de ceux qui ſe vantoient d'avoir aperçu ſes fautes, conſiſtoit ſeulement dans une application facile des Méthodes, que lui même leur avoit enſeignées, à des Théories dont il avoit applani d'avance les plus grandes difficultés.

Preſque toujours les hommes médiocres cherchent à ſe faire valoir par une ſévérité proportionnée à la haute idée qu'ils veulent donner de leur jugement, ou de leur génie. Inexorables pour tout ce qui s'éleve au deſſus d'eux, ils ne pardonnent même pas à l'inferiorité. On diroit qu'un ſentiment ſecret les avertit du beſoin qu'ils ont de rabaiſſer les autres. Au contraire, le premier mouvement de M. EULER le portoit à célébrer les talens dés l'iſtant où quelques eſſais heureux frappoient ſes regards, & ſans attendre qu'une juſte admiration eût ſollicité ſon ſuffrage. On le voit employer ſon tems à refaire, à éclaircir ſes ouvrages & même à réſoudre des problêmes déja réſolus, qui ne lui laiſſoient plus que le mérite de plus d'élégance & de méthode, avec la même ardeur, la même conſtance, qu'il eût miſes à pourſuivre une vérité nouvelle, dont la découverte auroit ajouté à ſa renommée. D'ailleurs, ſi le déſir ardent de la gloire eût exiſté au fond de ſon cœur, la franchiſe de ſon caractere ne lui eût pas permis d'en cacher les mouvemens. Mais cette gloire, dont il s'occupoit ſi peu, vint le chercher. La ſé-

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condité ſinguliére de ſon génie frappoit même ceux qui n'étoient pas en état d'entendre ſes ouvrages. Quoiqu'uniquement livré à la Géométrie, ſa réputation s'étendit parmi les hommes les plus étrangers à cette ſcience; & il fût pour l'Europe entiere, non ſeulement grand Géomêtre, mais un grand-homme.

Il eſt d'uſage en Ruſſie d'accorder des titres militaires à des hommes très étrangers au ſervice. C'eſt rendre hommage au préjugé qui faiſoit regarder cet Etat comme la ſeule profeſſion noble, & avouer en même tems qu'on en reconnoit toute la fauſſeté. Quelques ſavans ont obtenu juſqu'au grade de Général-Major. M. EULER n'en eût & n'en vouloit point avoir aucun. Mais quel titre pouvoit honorer le nom d'EULER? Et alors le reſpect pour la conſervation des droits naturels de l'homme, impoſe en quelque ſorte le devoir de donner l'exemple d'une ſage indifférence pour ces hochets de la vanité humaine, ſi puérile, mais ſi dangereux.

La plupart des Princes du Nord, dont il étoit perſonnellement connu, lui ont donné des marques de leur eſtime, ou plutôt de la vénération, qu'on ne pouvoit refuſer à la réunion d'une vertu ſi ſimple & d'un génie ſi vaſte & ſi élevé. Dans le voyage que le Prince Royal de Pruſſe fit à Petersbourg, il prévint la viſite de M. EULER & paſſa quelques heures à coté du lit de cet illuſtre vieillard, ayant ſes mains dans les ſiennes & tenant ſur ſes genoux un petit fils d'EU-

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d'EULER, que ſes diſpoſitions précoces pour la Géométrie avoient rendu l'objet particulier de ſa tendreſſe paternelle.

Tous les mathématiciens célebres qui exiſtent aujourd'hui ſont ſes éleves. Il n'en eſt aucun qui ne ſe ſoit formé par la lecture de ſes ouvrages, qui n'ait reçu de lui les formules, la métode qu'il emploie, qui, dans ſes découvertes ne ſoit guidé & ſoutenu par le génie d'EULER. Il doit cet honneur à la révolution qu'il a produite dans les ſciences mathématiques, en les ſoumettant toutes à l'analyſe; à ſa force pour le travail, qui lui a permis d'ambraſſer toute l'étendue de ces ſciences; à l'ordre méthodique qu'il a ſçu mettre dans ſes grands ouvrages; à la ſimplicité, à l'élégance de ſes formules; à la clarté de ſes méthodes & de ſes démonſtrations, qu'augmente encore la multiplicité & le choix de ſes exemples. Ni NEWTON, ni DESCARTES même, dont l'influence a été ſi puiſſante, n'ont obtenu cette gloire; & juſqu'ici, ſeul entre tous les Géomêtres, M. EULER l'a poſſédée toute entiere & ſans partage.

Mais comme Profeſſeur, il a formé des éleves qui lui appartiennent plus particulierement, & parmi les quels nous citerons ſon fils ainé, que l'Académie des ſciences a choiſi pour le remplacer, ſans craindre que cette ſucceſſion honorable accordée au nom d'EULER, connue à celui de BERNOULLI, pût devenir un exemple dangereux; un ſecond fils livré aujourd'hui à l'étude de la Médicine, mais qui dans ſa jeuneſſe

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a remporté dans cette Académie un prix ſur les altérations du moyen mouvement des planêtes: M. LEXELL, qu'une mort prématurée vient d'enlever aux ſciences, enfin M. FUSS, le plus jeune de ſes diſciples, le compagnon de ſes derniers travaux, qui, envoyé de BASLE à M. EULER, par M. DANIEL BERNOULLI, s'eſt montré digne par ſes ouvrages du choix de BERNOULLI & des leçons d'EULER & qui, aprés avoir rendu dans l'Académie de Petersbourg un hommage public à ſon illuſtre Maitre, vient de s'unir à ſa petite-fille.

De ſeize Profeſſeurs attachés à l'Académie de Petersbourg, huit avoient été formés par lui & tous connus par leurs ouvrages & décorés de titres académiques, ſe glorifioient de pouvoir y ajouter celui de diſciples d'EULER.

Il avoit conſervé toute ſa facilité, & en apparence toutes ſes forces. Aucun changement n'annonçoit que les ſciences fuſſent menacées des le perdre. Le ſept Septembre 1783, après s'être amuſé à calculer ſur une ardoiſe les loix du mouvement aſcenſionel des machines Aëroſtatiques, dont la découverte récente occupoit alors toute l'Europe, il dina avec M. LEXELL & ſa famille, parla de la planette d'Herſchell, & des calculs qui en déterminent l'orbite; peu de tems aprés, il fit venir ſon petit-fils, avec le quel il badinoit en prenant quelques taſſes de Thé, lorſque tout à coup la pipe, qu'il tenoit à la main, lui échappa, & il ceſſa de calculer & de vivre.

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Telle fut la fin d'un des hommes les plus grands & les plus extraordinaires que la nature ait jamais produits, dont le génie fut également capable des plus grands éfforts & du travail le plus continu, qui multiplia ſes productions au delà de ce qu'on eut oſé attendre des forces humaines & qui cependant fut original dans chacune, dont la tête fut toujours occupée & l'âme toujours calme, & qui, par une deſtinée malheureuſement trop rare, réunit & mérita de réunir un bonheur presque ſans nuage à une gloire qui ne fut jamais conteſtée.

Sa mort a été regardée comme une perte publique, même dans le pays qu'il habitoit. L'Académie de Petersbourg a porté ſolemnellement ſon deuil & lui a décerné à ſes frais un buſte de marbre, qui doit être placé dans ſes ſalles d'aſſemblée. Elle lui avoit deja rendu pendant ſa vie un honneur plus ſingulier peut-être. Dans un tableau allégorique, la figure de la Géométrie s'apuie ſur une planche chargée de calculs; & ce ſont les formules de ſa nouvelle Théorie de la lune, que l'Académie a ordonné d'y inſcrire. Ainſi un Pays qu'au commencement de ce ſiecle, nous regardions encore comme barbare, apprend aux nations les plus éclairées de l'Europe à honnorer la vie des grands hommes & leur mémoire récente. Il donne à ces nations un exemple que pluſieurs d'entre elles auroient à rougir peut-être, de n'avoir sçu ni prévenir, ni même imiter.

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Quid ſit Calculus Differentialis, atque in genere Analyſis infinitorum, iis qui nulla adhuc eius cognitione ſunt imbuti, vix explicari poteſt: neque hic, uti in aliis diſciplinis fieri ſolet, exordium tractationis a definitione commode ſumere licet. Non quod huius calculi nulla plane detur definitio; ſed quoniam ad eam intelligendam eiuſmodi opus eſt notionibus, non ſolum in vita communi, verum etiam in ipſa Analyſi finitorum minus uſitatis, quae demum in Calculi Differentialis pertractatione evolvi atque explicari ſolent: quo fit, ut eius definitio non ante percipi queat, quam eius principia iam ſatis dilucide fuerint perſpecta. Primum igitur hic calculus circa quantitates variabiles verſatur: etſi enim omnis quantitas ſua natura in infinitum augeri & diminui poteſt; tamen dum calculus ad certum quoddam inſtitutum dirigitur, aliae quantitates conſtanter eandem magnitudinem retinere concipiuntur, aliae vero per omnes gradus auctionis ac diminutionis variari: ad quam diſtinctionem notandam illae quantitates conſtantes, hae vero variabiles vocari ſolent; ita ut hoc diſcrimen non tam in rei natura, quam in quaeſtionis, ad quam calculus refertur, indole ſit poſitum. Quoniam haec differentia inter quantitates conſtantes & variabiles exemplo maxime illuſtrabitur, conſideremus iactum globi ex tormento bellico vi pulveris pyrii exploſi; ſiquidem hoc exemplum ad rem dilucidandam imprimis idoneum videtur. Plures igitur hic occurrunt quantitates, quarum ratio in iſta inveſtigatione eſt habenda: primo ſcilicet quantitas pulveris pyrii; tum elevatio tormenti ſupra horizontem; tertio longitudo iactus ſuper plano horizontali; quarto tempus, quo globus exploſus in aere verſatur: ac niſi experimenta eodem tormento inſtituantur, inſuper eius longitudo cum pondere globi in computum trahi deberet. Verum hic a

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varietate tormenti & globi animum removeamus, ne in quaeſtiones nimium implicatas incidamus. Quodſi ergo ſervata perpetuo eadem pulveris pyrii quantitate, elevatio tormenti continuo immutetur, iactuſque longitudo cum tempore tranſitus globi per aerem requiratur; in hac quaeſtione copia pulveris ſeu vis impulſus erit quantitas conſtans, elevatio autem tormenti cum longitudine iactus eiuſque duratione ad quantitates variabiles referri debebunt; ſi quidem pro omnibus elevationis gradibus has res definire velimus, ut inde innoteſcat, quantae mutationes in longitudine ac duratione iactus ab omnibus elevationis variationibus oriantur. Alia autem erit quaeſtio, ſi ſervata eadem tormenti elevatione, quantitas pulveris pyrii continuo mutetur, & mutationes, quae inde in iactum redundant, definiri debeant: hic enim elevatio tormenti erit quantitas conſtans, contra vero quantitas pulveris pyrii, & longitudo ac duratio iactus quantitates variabiles. Sic igitur patet, quomodo mutato quaeſtionis ſtatu eadem quantitas modo inter conſtantes, modo inter variabiles numerari queat: ſimul autem hinc intelligitur, ad quod in hoc negotio maxime eſt attendendum, quomodo quantitates variabiles aliae ab aliis ita pendeant, ut mutata una reliquae neceſſario immutationes recipiant. Priori ſcilicet caſu, quo quantitas pulveris pyrii eadem manebat, mutata tormenti elevatione etiam longitudo & duratio iactus mutantur; ſuntque ergo longitudo & duratio iactus quantitates variabiles pendentes ab elevatione tormenti, hacque mutata ſimul certas quaſdam mutationes patientes: poſteriori vero caſu pendent a quantitate pulveris pyrii, cuius mutatio in illis certas mutationes producere debet. Quae autem quantitates hoc modo ab aliis pendent, ut his mutatis etiam ipſae mutationes ſubeant, eae harum functiones appellari ſolent; quae denominatio latiſſime patet, atque omnes modos, quibus una quantitas per alias determinari poteſt, in ſe complectitur. Si igitur x *denotet quantitatem variabilem, omnes quantitates, quae utcunque ab* x *pendent, ſeu per eam determinantur, eius functiones vocantur; cuiuſ-*

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modi ſunt quadratum eius xx, *aliaeve potentiae quaecunque nec non quantitates ex his utcunque compoſitae; quin etiam tranſcendentes, & in genere quaecunque ita ab* x *pendent, ut aucta vel diminuta* x *ipſae mutationes recipiant. Hinc, iam naſcitur quaeſtio, qua quaeritur, ſi quantitas* x *data quantitate ſive augeatur ſive diminuatur, quantum inde quaevis eius functiones immutentur, ſeu quantum incrementum decrementumve accipiant. Caſibus quidem ſimplicioribus haec quaeſtio facile reſolvitur: ſi enim quantitas* x *augeatur quantitate* ω, *eius quadratum* xx *hinc incrementum capiet* 2xω + ωω; *ſicque incrementum ipſius* x *ſe habebit ad incrementum ipſius* xx, *ut* ω *ad* 2xω + ωω, *hoc eſt, ut* 1 *ad* 2x + ω; *ſimilique modo in aliis caſibus ratio incrementi ipſius* x *ad incrementum, vel decrementum, quod quaevis eius functio inde adipiſcitur, conſiderari ſolet. Eſt vero inveſtigatio rationis huiuſmodi incrementorum ipſa non ſolum maximi momenti, ſed ei etiam univerſa Analyſis infinitorum innititur. Quod quo clarius appareat, ſumamus exemplum ſuperius quadrati* xx, *cuius incrementum* 2xω + ωω, *quod capit, dum ipſa quantitas* x *incremento* ω *augetur, vidimus ad hoc rationem tenere, ut* 2x + ω *ad* 1; *unde perſpicuum eſt, quo minus ſumatur incrementum* ω, *eo propius iſtam rationem accedere ad rationem* 2x *ad* 1; *neque tamen ante prorſus in hanc rationem abit, quam incrementum illud* ω *plane evaneſcat. Hinc intelligimus, ſi quantitatis variabilis* x *incrementum* ω *in nihilum abeat, tum etiam quadrati eius* xx *incrementum inde oriundum quidem evaneſcere, verumtamen ad id rationem tenere ut* 2x *ad* 1; & *quod hic de quadrato eſt dictum, de omnibus aliis functionibus ipſius* x *eſt intelligendum; quippe quarum incrementa evaneſcentia, quae capiunt, dum ipſa quantitas* x *incrementum evaneſcens ſumit ad hoc ipſum certam & aſſignabilem rationem tenebunt. Atque hoc modo ſumus deducti ad definitionem* calculi Differentialis, *qui eſt* methodus determinandi rationem incrementorum evaneſcentium, quae functiones quaecunque accipiunt,

dum quantitati variabili, cuius ſunt functiones, incrementum evaneſcens tribuitur: *hacque definitione veram indolem calculi differentialis contineri, atque adeo exhauriri, iis, qui in hoc genere non ſunt hoſpites, facile erit perſpicuum. Calculus igitur differentialis non tam in his ipſis incrementis evaneſcentibus, quippe quae ſunt nulla, exquirendis, quam in eorum ratione ac proportione mutua ſcrutanda occupatur: & cum hae rationes finitis quantitatibus exprimantur, etiam hic calculus circa quantitates finitas verſari eſt cenſendus. Quamvis enim praecepta, uti vulgo tradi ſolent, ad iſta incrementa evaneſcentia definienda videantur accommodata; nunquam tamen ex iis abſolute ſpectatis, ſed potius ſemper ex eorum ratione concluſiones deducuntur. Simili vero modo calculi integralis ratio eſt comparata, qui convenientiſſime ita definitur, ut dicatur eſſe* methodus ex cognita ratione incrementorum evaneſcentium ipſas illas functiones, quarum ſunt incrementa, inveniendi. *Quo autem facilius hae rationes colligi, atque in calculo repraeſentari poſſint, haec ipſa incrementa evaneſcentia, etiamſi ſint nulla, tamen certis ſignis denotari ſolent; quibus adhibitis nihil obſtat, quo minus iis certa nomina imponantur. Vocantur itaque differentialia, quae cum quantitate deſtituantur, infinite parva quoque dicuntur; quae igitur ſua natura ita ſunt interpretanda, ut omnino nulla ſeu nihilo aequalia reputentur. Ita ſi quantitati* x *incrementum tribuatur* ω, *ut abeat in* $x+\omega$, *eius quadratum* xx *abibit in* $xx+2x\omega+\omega\omega$, *ideoque incrementum capit* $2x\omega+\omega\omega$; *quare incrementum ipſius* x, *quod eſt* ω, *ſe habebit ad incrementum quadrati, quod eſt* $2x\omega+\omega\omega$, *uti* 1 *ad* $2x+\omega$; *quae ratio abit in* 1 *ad* $2x$, *tum demum, cum* ω *evaneſcit. Fiat igitur* $\omega=0$, *& ratio iſtorum incrementorum evaneſcentium, quae ſola in calculo differentiali ſpectatur, utique eſt ut* 1 *ad* $2x$; *neque viciſſim haec ratio veritati eſſet conſentanea, niſi revera illud incrementum* ω *evaneſceret, penituſque nihilo fieret aequale. Quodſi ergo hoc nihilum per* ω *indicatum referat incrementum quantitatis* x, *quia hoc ſe ha-*

habet ad incrementum quadrati xx *ut* 1 *ad* 2x, *erit quadrati* xx *incrementum* $= 2x\omega$, *ideoque etiam nihilo aequale; unde ſimul conſtat annihilationem horum incrementorum non obſtare, quominus eorum ratio, quae eſt ut* 1 *ad* 2x *ſit determinata. Quod nihilum iam hic littera* ω *exhibetur, id in calculo differentiali, quia ut incrementum quantitatis* x *ſpectatur, ſigno* dx *repraeſentari, eiuſque differentiale vocari ſolet; poſitoque* dx *loco* ω, *ipſius* xx *differentiale erit* 2xdx. *Simili modo oſtenditur fore cubi* x^3 *differentiale* $= 3xxdx$, *& in genere cuiuſque dignitatis* x^n *differentiale fore* $= nx^{n-1}dx$. *Quaecunque autem aliae functiones ipſius* x *proponantur, in calculo differentiali regulae traduntur eorum differentialia inveniendi: verum perpetuo tenendum eſt, cum haec differentialia abſolute ſint nihila, ex iis nihil aliud concludi, niſi eorum rationes mutuas, quae utique ad quantitates finitas reducuntur. Cum autem hoc modo, qui ſolus eſt rationi conſentaneus, principia Calculi differentialis ſtabiliuntur, omnes obtrectationes; quae contra hunc calculum proferri ſunt ſolitae, ſponte corruunt; quae tamen ſummam vim retinerent, ſi differentialia ſeu infinite parva non plane annihilarentur. Pluribus autem, qui Calculi differentialis praecepta tradidere, viſum eſt differentialia a nihilo abſoluto ſecernere, peculiaremque ordinem quantitatum infinite parvarum, quae non penitus evaneſcant, ſed quantitatem quandam, quae quidem eſſet omni aſſignabili minor, retineant, conſtituere: his igitur iure eſt obiectum, rigorem geometricum negligi, & concluſiones inde deductas, propterea quod huiuſmodi infinite parva negligerentur, merito eſſe ſuſpectas: quantumvis enim exigua haec infinitae parva concipiantur, tamen non ſolum ſingulis, ſed etiam pluribus atque adeo innumerabilibus ſimul reiiciendis, errorem tandem inde enormem reſultare poſſe. Quam obiectionem perperam eiuſmodi exemplis, quibus per calculum differentialem eaedem concluſiones ac per Geometriam elementarem eliciuntur, infringere conantur: nam ſi ea infinite parva, quae in calculo negliguntur, non ſunt nihil, inde neceſ-*

ſario error, iſque eo maior, quo magis ea coacervantur, reſultare debet; hocque ſi minus eveniat, id potius vitio calculi, quo nonnunquam errores per alios errores compenſantur, eſſet tribuendum, quam ipſe calculus ab erroris ſuſpicione liberaretur. Quodſi autem nullo novo errore huiuſmodi compenſatio fiat, talibus exemplis luculenter id ipſum, quod volo, evincitur, ea quae fuerint neglecta, omnino & abſolute pro nihilo eſſe habenda; neque infinite parva, quae in calculo differentiali tractantur, a nihilo abſoluto diſcrepare. Minime etiam negotium conficitur, quando a nonnullis infinite parva ita deſcribuntur, ut inſtar pulviſculorum reſpectu vaſti montis vel etiam totius globi terreſtris ſpectari debeant: etſi enim qui magnitudinem totius globi terreſtris calculo determinare ſuſceperit, ei error non unius ſed plurium millium pulviſculorum facile condonari ſoleat; tamen rigor geometricus etiam a tantillo errore abhorret, nimiſque gravis eſſet haec obiectio, ſi ullam vim retineret. Deinde etiam difficile dictu eſt, quid lucri inde ſperent, qui infinite parva a nihilo diſtingui volunt: metuunt autem, ne, ſi plane evaneſcant, etiam comparatio eorum, ad quam totum negotium perduci ſentiunt, tollatur: quomodo enim abſolute nihila inter ſe comparari queant, nullo modo concipi poſſe profitentur. Neceſſe ergo putant iis aliquam magnitudinem relinquere, quo habeant aliquid, in quo comparationem inſtituant: hanc tamen magnitudinem tam parvam admittere coguntur, ut quaſi eſſet nulla, ſpectari ac ſine errore in calculo negligi poſſit. Neque tamen certam ac definitam ipſi magnitudinem, licet incomprehenſibiliter parvam, aſſignare audent; ſemper enim ſi eam bis terve minorem aſſumerent, eodem modo comparationes ſe eſſent habiturae. Ex quo perſpicuum eſt, nihil plane ipſam magnitudinem ad comparationem inſtituendam conferre, hancque adeo non tolli, etiamſi illa magnitudo penitus evaneſcat. Ex dictis autem ſupra manifeſtum eſt, eam comparationem, quae in calculo differentiali ſpectatur, ne locum quidem habere, niſi illa incrementa prorſus evaneſcant: incrementum enim quantita-

titatis x, *quod in genere indicavimus per* ω, *ad incrementum quadrati* xx, *quod eſt* $2x\omega+\omega\omega$, *rationem habet ut* 1 *ad* $2x+\omega$; *quae ſemper differt a ratione* 1 *ad* 2x, *niſi ſit* $\omega=0$; *at ſi ſtatuamus eſſe* $\omega=0$, *tum demum vere affirmare poſſumus; hanc rationem fieri exacte ut* 1 *ad* 2x. *Interim tamen perſpicitur, quo minus illud incrementum* ω *accipiatur, eo propius ad hanc rationem accedi; unde non ſolum licet, ſed etiam naturae rei convenit, haec incrementa primum ut finita conſiderare, atque etiam in figuris, ſi quibus opus eſt ad rem illuſtrandam, finite repraeſentare; deinde vero haec incrementa cogitatione continuo minora fieri concipiantur, ſicque eorum ratio continuo magis ad certum quendam limitem appropinquare reperietur, quem autem tum demum attingant, cum plane in nibilum abierint. Hic autem limes, qui quaſi rationem ultimam incrementorum illorum conſtituit, verum eſt obiectum Calculi differentialis; cuius igitur prima fundamenta is ieciſſe exiſtimandus eſt, cui primum in mentem venit, has rationes ultimas, ad quas quantitatum variabilium incrementa dum continuo magis diminuuntur, appropinquant, & cum evaneſcunt, tum demum attingunt, contemplari. Huius autem ſpeculationis veſtigia deprehendimus apud antiquiſſimos Auctores, quibus idcirco idea quaedam leviſque cognitio Analyſis infinitorum abiudicari nequit. Paullatim deinde haec ſcientia maiora accepit incrementa, neque ſubito ad id faſtigium, in quo nunc cernitur, eſt evecta; etiamſi quidem in ea multo plura adhuc ſint occulta, quam in lucem protracta. Cum enim Calculus differentialis ad omnis generis functiones, utcunque ſint compoſitae, extendatur, non repente methodus innotuit, omnium plane functionum incrementa evaneſcentia inter ſe comparandi; ſed ſenſim haec inventio ad functiones continuo magis complicatas proceſſit. Quod ſcilicet ad functiones rationales attinet, ratio ultima, quam earum incrementa evaneſcentia inter ſe tenent, multo ante* NEUTONI *ac* LEIBNIZII *tempora aſſignari potuit; ita ut Calculus differentialis, quatenus ad ſolas functiones rationales applicatur, diu ante haec tem-*

pora

pora inventus ſit cenſendus. Tum vero nullum eſt dubium, quin NEUTONO *eam calculi differentialis partem, quae circa functiones irrationales verſatur, acceptam referre debeamus; ad quam inſigni ſuo Theoremate de evolutione generali poteſtatum binomii feliciter eſt deductus, quo eximio invento limites calculi differentialis iam mirifice erant amplificati.* LEIBNIZIO *autem non minus ſumus obſtricti, quod hunc calculum, antehac tantum velut ſingulare artificium ſpectatum, in formam diſciplinae redegerit, eiuſque praecepta tanquam in ſyſtema collegerit, ac dilucide explicaverit. Hinc enim maxima ſubſidia ſuggerebantur, ad hunc calculum ulterius excolendum, & ea, quae adhuc deſiderabantur, ex certis principiis elicienda. Mox igitur ſtudio cum ipſius* LEIBNIZII, *tum* BERNOUILLIORUM *ad hoc ab eo incitatorum, fines Calculi differentialis etiam ad functiones tranſcendentes, quae pars adhuc fuerat inculta, ſunt promoti, tum vero etiam ſolidiſſima fundamenta Calculi integralis conſtituta; quibus inſiſtentes, qui deinceps in hoc genere elaborarunt, continuo maiora incrementa addiderunt.* NEUTONUS *vero etiam ampliſſima dederat ſpecimina Calculi integralis, cuius prima inventio, cum a prima origine calculi differentialis vix ſeparari queat, non ita abſolute conſtitui poteſt; & quoniam maxima eius pars adhuc excolenda reſtat, hic calculus ne nunc quidem pro abſolute invento haberi poteſt; ſed potius quantum cuique pro viribus ad eius perfectionem conferre contigerit, id grata mente agnoſcere debemus. Atque haec de gloria inventionis huius calculi tenenda eſſe iudico, de qua quidem antehac tantopere eſt diſceptatum. Quod autem ad varia nomina, quae iſti calculo a diverſarum nationum Mathematicis imponi ſolent, attinet, ea omnia huc redeunt, ut cum data hic definitione egregie conſentiant: ſive enim incrementa illa evaneſcentia, quorum ratio conſideratur, differentialia vocentur, ſive fluxiones, ea ſemper nihilo aequalia ſunt intelligenda; in quo vera notio infinite parvorum conſtitui debet. Hinc vero etiam omnia, quae de differentialibus*

ſecun-

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ſecundi & altiorum ordinum curioſe magis quam utiliter ſunt diſputata, reddentur planiſſima, cum omnia per ſe aeque evaneſcant neque ea unquam per ſe, ſed potius eorum relatio mutua ſpectari ſoleat. Cum enim ratio, quam duarum functionum incrementa evaneſcentia tenent, iterum per functionem quandam exprimatur, ſi & huius functionis incrementum evaneſcens cum aliis conferatur, res ad differentialia ſecunda referri eſt cenſenda; ſicque porro progreſſio ad differentialia altiorum graduum intelligi debet, ita ut ſemper quantitates finitae revera animo obverſentur, ſignaque differentialium tantum ad eas commode repraeſentandas adhibeantur. Primo quidem intuitu iſta Analyſis infinitorum deſcriptio pleriſque levis ac nimis ſterilis videtur, etſi ſpecies illa arcana infinite parvorum re haud plus polliceatur: verum ſi rationes, quae inter incrementa evaneſcentia functionum quarumvis intercedunt, probe cognoſcamus, haec cognitio ſaepenumero per ſe maximi eſt momenti; tum vero in pleriſque iiſque maxime arduis inveſtigationibus ita eſt neceſſaria, ut ſine eius adminiculo nihil plane intelligi poſſit. Veluti ſi quaeſtio ſit de motu globi ex tormento exploſi, ſimulque ratio reſiſtentiae aeris haberi debeat, quomodo motus per ſpatium finitum ſit futurus, nullo modo ſtatim definire licet, dum tam directio ſemitae, in qua globus incedit, quam ipſius celeritas, a qua reſiſtentia pendet, quovis momento immutatur. Quo minus autem ſpatium, per quod motus fiat, conſideremus, eo minor erit illa variabilitas, eoque facilius ad cognitionem veri pertingere licebit; quodſi autem illud ſpatium plane evaneſcens reddamus, quia iam omnis inaequalitas tam in directione viae quam in celeritate tollitur, effectum reſiſtentiae per regulas motus accurate definire, motuſque mutationem puncto temporis productam aſſignare licebit. Cognitis autem his mutationibus momentaneis, ſeu potius cum ipſae ſint nullae, earum relatione mutua, iam plurimum ſumus lucrati; atque calculi integralis opus eſt, exinde motum per ſpatium finitum variatum concludere. Minime autem neceſſe

eſſe

eſſe arbitror uſum Calculi differentialis atque Analyſeos infinitorum in genere pluribus oſtendere; cum nunc quidem ſatis ſit exploratum, ſi vel leviſſimam inveſtigationem, in quam motus corporum tam ſolidorum quam fluidorum ingrediatur, accuratius inſtituere velimus, id non ſolum non ſine Analyſi infinitorum praeſtari poſſe, ſed hanc ipſam ſcientiam ſaepe nondum ſatis excultam eſſe, ut rem penitus explicare valeamus. Per omnes ſcilicet Matheſeos partes uſus huius Analyſeos ſublimioris uſque adeo diffunditur, ut omnia, quae ſine eius interventu adhuc expedire licuit, pro nihilo propemodum ſint habenda.

Conſtitui igitur in hoc libro univerſum Calculum differentialem ex veris principiis derivare, atque ita copioſe pertractare, ut nihil praetermitterem eorum, quae quidem adhuc eo pertinentia ſunt inventa. In duas opus diviſi partes, in quarum priori iactis calculi differentialis fundamentis methodum expoſui omnis generis functiones differentiandi, neque tantum differentialia primi ordinis, ſed etiam ſuperiorum ordinum inveniendi; ſive functiones unicam variabilem ſive duas pluresve involvant. In altera autem parte ampliſſimum huius calculi uſum in ipſa Analyſi finitorum ac doctrina ſerierum expoſui; ubi etiam imprimis Theoriam maximorum ac minimorum dilucide explicavi. De uſu autem huius calculi in Geometria linearum curvarum nihil adhuc affero, quod eo minus deſiderabitur, cum in aliis operibus haec pars ita copioſe ſit pertractata, ut adeo prima calculi differentialis principia quaſi ex Geometria ſint petita, ad hancque ſcientiam, cum vix ſatis eſſent evoluta, ſumma cura applicata. Hic autem omnia ita intra Analyſeos purae limites continentur, ut ne ulla quidem figura opus fuerit, ad omnia huius calculi praecepta explicanda.

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INSTITUTIONUM
CALCULI DIFFERENTIALIS

PARS PRIOR

CONTINENS

COMPLETAM HUIUS CALCULI

EXPLICATIONEM

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CAPUT I.

DE DIFFERENTIIS FINITIS.

I.

EX iis, quae in Libro ſuperiori de quantitatibus variabilibus atque functionibus ſunt expoſita, perſpicuum eſt, prout quantitas variabilis actu variatur, ita omnes eius functiones variationem pati. Sic, ſi quantitas variabilis x capiat incrementum ω, ita ut pro x ſcribatur $x + \omega$, omnes functiones ipſius x, cuiuſmodi ſunt xx; x^3; $\frac{a + x}{xx + aa}$, alios induent valores: ſcilicet xx abibit in $xx + 2x\omega + \omega\omega$; x^3 abibit in $x^3 + 3xx\omega + 3x\omega\omega + \omega^3$; & $\frac{a + x}{aa + xx}$ tranſmutabitur in $\frac{a + x + \omega}{aa + xx + 2x\omega + \omega\omega}$.

Huiuſmodi ergo alteratio ſemper orietur, niſi functio ſpeciem tantum quantitatis variabilis mentiatur, revera autem ſit quantitas conſtans, veluti x^0: quo caſu talis functio invariata manet, utcunque quantitas x immutetur.

2. Quae cum ſint ſatis expoſita, propius accedamus ad eas functionum affectiones, quibus univerſa analyſis infinitorum innititur. Sit igitur y functio quaecunque quantitatis variabilis x: pro qua ſucceſſive valores in arithmetica progreſſione procedentes ſubſtituantur, ſcilicet: x; $x+\omega$; $x+2\omega$; $x+3\omega$; $x+4\omega$; &c. ac denotet y^{I} valorem quem functio y induit, ſi in ea loco x ſubſtituatur $x+\omega$; ſimili modo ſit y^{II} is ipſius y valor, ſi loco x ſcribatur $x+2\omega$; parique ratione denotent y^{III}; y^{IV}; y^{V}; &c. valores ipſius y, qui emergunt dum loco x ponuntur $x+3\omega$; $x+4\omega$; $x+5\omega$; &c. ita ut iſti diverſi valores ipſarum x & y ſequenti modo ſibi reſpondeant:

$$x;\ x+\omega;\ x+2\omega;\ x+3\omega;\ x+4\omega;\ x+5\omega;\ \&c.$$
$$y;\ y^{I};\ y^{II};\ y^{III};\ y^{IV};\ y^{V};\ \&c.$$

3. Quemadmodum ſeries arithmetica x; $x+\omega$; $x+2\omega$; &c. in infinitum continuari poteſt; ita ſeries ex functione y orta y; y^{I}; y^{II}; &c. quoque in infinitum progredietur, eiusque natura pendebit ab indole functionis y. Sic, ſi fuerit $y=x$; vel $y=ax+b$; ſeries y; y^{I}; y^{II}; &c. quoque erit arithmetica: ſi fuerit $y=\frac{a}{bx+c}$, ſeries prodibit harmonica: ſi autem ſit $y=a^x$, habebitur ſeries geometrica. Neque ulla excogitari poteſt ſeries, quae non hoc modo ex certa functione ipſius x oriri queat; vocari autem ſolet huiuſmodi functio ipſius x, ratione ſeriei, quae ex illa oritur, eius TERMINUS GENERALIS; quare cum omnis ſeries certa lege formata habeat terminum generalem, ea viciſſim ex certa ipſius x functione oritur, uti in doctrina de ſeriebus fuſius explicari ſolet.

4.

4. Hic autem potiſſimum ad differentias, quibus termini ſeriei y, y^{I}, y^{II}, y^{III}, &c. inter ſe diſcrepant, attendimus; quas ut ad differentialium naturam accomodemus, ſequentibus ſignis indicemus, ut ſit

$y^{I} - y = \Delta y$; $y^{II} - y^{I} = \Delta y^{I}$; $y^{III} - y^{II} = \Delta y^{II}$; &c.

Exprimet ergo Δy incrementum, quod functio y capit, ſi in ea loco x ponatur $x + \omega$, denotante ω numerum quemcunque pro lubitu aſſumtum. In doctrina quidem ſerierum ſumi ſolet $\omega = 1$; verum hic ad noſtrum inſtitutum expedit, valore generali uti, qui pro arbitrio augeri diminuive queat. Vocari quoque ſolet hoc incrementum Δy functionis y eius DIFFERENTIA, qua ſequens valor y^{I} primum y ſuperat, atque perpetuo tanquam incrementum conſideratur; etiamſi ſaepius re vera decrementum exhibeat, id quod ex eius valore negativo agnoſcitur.

5. Quoniam y^{II} oritur ex y, ſi loco x ſcribatur $x + 2\omega$; manifeſtum eſt eandem quantitatem eſſe orituram ſi primum pro x ponatur $x + \omega$, tumque denuo $x + \omega$ loco x ſtatuatur. Hinc y^{II} orietur ex y^{I}, ſi in hoc loco x ſcribatur $x + \omega$; eritque ideo Δy^{I} incrementum ipſius y^{I} quod capit poſito $x + \omega$ loco x; ſicque Δy^{I} vocatur ſimili modo *Differentia* ipſius y^{I}. Pari ratione porro erit Δy^{II} differentia ipſius y^{II}, ſeu eius incrementum, quod accipit, ſi loco x ponatur $x + \omega$; atque Δy^{III} erit differentia, ſeu incrementum ipſius y^{III}, & ita porro. Hoc pacto ex ſerie valorum ipſius y, qui ſunt y; y^{I}; y^{II}; y^{III}; &c. obtinebitur ſeries differentiarum Δy; Δy^{I}; Δy^{II}; &c. quae inveniuntur, ſi quilibet terminus illius ſeriei a ſequente ſubtrahatur.

6. Inventa ſerie differentiarum, ſi ex ea denuo differentiae capiantur, quamlibet a ſequente ſubtrahendo, orientur differentiae differentiarum, quae vocantur *Differentiae ſecundae*; hocque modo per characteres convenientiſſime repraeſentantur, ut ſignificet:

$$\Delta\Delta y = \Delta y^{I} - \Delta y$$
$$\Delta\Delta y^{I} = \Delta y^{II} - \Delta y^{I}$$
$$\Delta\Delta y^{II} = \Delta y^{III} - \Delta y^{II}$$
$$\Delta\Delta y^{III} = \Delta y^{IV} - \Delta y^{III}$$
&c.

Vocatur itaque $\Delta\Delta y$ differentia ſecunda ipſius y; $\Delta\Delta y^{I}$ differentia ſecunda ipſius y^{I}, & ita porro. Simili autem modo ex differentiis ſecundis, ſi denuo earum differentiae capiantur, prodibunt differentiae tertiae hoc modo ſcribendae $\Delta^{3}y$; $\Delta^{3}y^{I}$; &c. hincque porro differentiae quartae $\Delta^{4}y$; $\Delta^{4}y^{I}$; &c. ſicque ultra quouſque libuerit.

7. Repraeſentemus ſingulas has differentiarum ſeries ita in ſchemate, quo earum nexus facilius in oculos incidat:

PROGRESSIO ARITHMETICA.

x; $x+\omega$; $x+2\omega$; $x+3\omega$; $x+4\omega$; $x+5\omega$; &c.

VALORES FUNCTIONIS.

y; y^{I}; y^{II}; y^{III}; y^{IV}; y^{V}; &c.

DIFFERENTIAE PRIMAE.

Δy; Δy^{I}; Δy^{II}; Δy^{III}; Δy^{IV}; &c.

DIFF. II. $\Delta\Delta y$; $\Delta\Delta y^{I}$; $\Delta\Delta y^{II}$; $\Delta\Delta y^{III}$; &c.

DIFF. III. $\Delta^{3}y$; $\Delta^{3}y^{I}$; $\Delta^{3}y^{II}$; &c.

DIFF. IV. $\Delta^{4}y$; $\Delta^{4}y^{I}$; &c.

DIFF. V. $\Delta^{5}y$; &c.

&c.

quarum quaelibet ex praecedente oritur, quoſque terminos a ſequentibus ſubtrahendo. Quacunque ergo functione ipſius x loco y ſubſtituta, quoniam valores y^{I}, y^{II}, y^{III}, &c. per no-

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notas compoſitiones facile formantur, ex iis ſine labore ſingulae differentiarum ſeries invenientur.

8. Ponamus eſſe $y = x$; eritque $y^{I} = x^{I} = x + \omega$; $y^{II} = x^{II} = x + 2\omega$: & ita porro. Unde differentiis ſumendis erit $\Delta x = \omega$; $\Delta x^{I} = \omega$; $\Delta x^{II} = \omega$; &c. ideoque omnes differentiae primae ipſius x erunt conſtantes, ac proinde differentiae ſecundae omnes evaneſcent; pariterque differentiae tertiae, & ſequentium ordinum omnes. Cum igitur ſit $\Delta x = \omega$, ob analogiam loco litterae ω iſte character Δx commode adhibebitur. Quantitatis ergo variabilis x, cuius valores ſucceſſivi x, x^{I}, x^{II}, x^{III}, &c. arithmeticam progreſſionem conſtituere aſſumuntur, differentiae Δx, Δx^{I}, Δx^{II}, &c. erunt conſtantes atque inter ſe aequales; ac propterea erit $\Delta\Delta x = 0$, $\Delta^{3} x = 0$, $\Delta^{4} = 0$, ſicque porro

9. Pro valoribus ipſius x, qui ipſi ſucceſſive tribuuntur, progreſſionem arithmeticam hic aſſumſimus, ita ut horum valorum differentiae primae ſint conſtantes, ſecundae ac reliquae omnes evaneſcant. Quod etſi ab arbitrio noſtro pendet, cum aliam quamcunque progreſſionem, aeque adhibere potuiſſemus; tamen progreſſio arithmetica prae reliquis omnibus commodiſſime uſurpari ſolet, cum quod ſit ſimpliciſſima atque intellectu facillima, tum vero maxime, quod ad omnes omnino valores, quos quidem x induere poteſt, pateat. Tribuendo enim ipſi ω valores tam negativos quam affirmativos, in hac ſerie valorum ipſius x omnes omnino continentur quantitates reales, quae in locum ipſius x ſubſtitui poſſunt: contra autem ſi ſeriem geometricam elegiſſemus, ad valores negativos nullus aditus patuiſſet. Hanc ob cauſam variabilitas functionum y ex valoribus ipſius x progreſſionem arithmeticam conſtituentibus aptiſſime diiudicatur.

10. Uti eſt $\Delta y = y^{I} - y$, ita differentiae ulteriores quoque ex terminis primae ſeriei y, y^{I}, y^{II}, y^{III}, &c. definiri poſſunt.

Cum

Cum enim ſit $\Delta y^{I} = y^{II} - y^{I}$
erit
$$\Delta\Delta y = y^{II} - 2y^{I} + y$$
&
$$\Delta\Delta y^{I} = y^{III} - 2y^{II} + y^{I}$$
ideoque
$$\Delta^{3} y = \Delta\Delta y^{I} - \Delta\Delta y = y^{III} - 3y^{II} + 3y^{I} - y$$
ſimili modo erit
$$\Delta^{4} y = y^{IV} - 4y^{III} + 6y^{II} - 4y^{I} + y$$
&
$$\Delta^{5} y = y^{V} - 5y^{IV} + 10y^{III} - 10y^{II} + 5y^{I} - y$$
quarum formularum coefficientes numerici eandem legem tenent, quae in poteſtatibus Binomii obſervatur. Quemadmodum ergo differentia prima ex duobus terminis ſeriei y; y^{I}; y^{II}; y^{III}; &c. determinatur, ita differentia ſecunda determinatur ex tribus, tertia ex quatuor, & ita de ceteris. Cognitis autem differentiis cuiuſque ordinis ipſius y, ſimili modo differentiae omnium ordinum ipſius y^{I}; y^{II}; &c. definientur.

11. Propoſita ergo quacunque Functione y ſingulae eius differentiae, tam prima, quam ſequentes, quae quidem differentiae ω, qua valores ipſius x progrediuntur, reſpondent, poterunt inveniri. Neque vero ad hoc opus eſt, ut ſeries valorum ipſius y ulterius continuetur; quemadmodum enim differentia prima Δy reperitur, ſi in y loco x ſcribatur $x + \omega$, atque a valore orto y^{I} ipſa functio y ſubtrahatur; ita differentia ſecunda $\Delta\Delta y$ obtinebitur ſi in differentia prima Δy loco x ponatur $x + \omega$, ut oriatur Δy^{I}, atque Δy a Δy^{I} ſubtrahatur. Simili modo ſi differentiae ſecundae $\Delta\Delta y$ capiatur differentia, eam ſubtrahendo a valore, quem induit, ſi loco x ponatur $x + \omega$, proveniet differentia tertia $\Delta^{3} y$; hincque porro eodem modo differentia quarta $\Delta^{4} y$, &c. Dummodo ergo quis noverit differentiam primam cuiuſque functionis inveſtigare, ſimul poterit differentiam ſecundam, tertiam,

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tiam, omneſque ſequentes invenire, propterea quod differentia ſecunda ipſius y nil aliud eſt, niſi differentia prima ipſius Δy; & differentia tertia ipſius y nil aliud, niſi differentia prima ipſius $\Delta\Delta y$; ſicque porro de reliquis.

12. Si functio y fuerit ex duabus pluribuſve partibus compoſita, ut ſit $y = p + q + r +$ &c.; tum quia eſt $y^I = p^I + q^I + r^I +$ &c., erit differentia $\Delta y = \Delta p + \Delta q + \Delta r +$ &c., ſimilique modo porro $\Delta\Delta y = \Delta\Delta p + \Delta\Delta q + \Delta\Delta r +$ &c., unde inventio differentiarum, ſi functio propoſita ex partibus fuerit compoſita, non parum facilior redditur. Quod ſi vero functio y fuerit productum ex duabus functionibus p & q, nempe $y = pq$, quia erit $y^I = p^I q^I$, & $p^I = p + \Delta p$, atque $q^I = q + \Delta q$, fiet $p^I q^I = pq + p\Delta q + q\Delta p + \Delta p\,\Delta q$, hincque $\Delta y = p\Delta q + q\Delta p + \Delta p\,\Delta q$. Unde, ſi ſit p quantitas conſtans $= a$, ob $\Delta a = 0$; erit functionis $y = aq$, differentia prima $\Delta y = a\Delta q$, ſimilique modo differentia ſecunda $\Delta\Delta y = a\Delta\Delta q$, tertia $\Delta^3 q = a\Delta^3 q$, & ita porro.

13. Quoniam omnis functio rationalis integra eſt aggregatum ex aliquot poteſtatibus ipſius x; omnes differentias functionum rationalium integrarum invenire poterimus, ſi differentias poteſtatum tantum exhibere noverimus. Hancobrem ſingularum poteſtatum quantitatis variabilis x differentias inveſtigemus in ſequentibus exemplis.

Cum autem ſit $x^0 = 1$, erit $\Delta x^0 = 0$; propterea quod x^0 non variatur, etiamſi x abeat in $x + \omega$.

Tum vero vidimus eſſe $\Delta x = \omega$; & $\Delta\Delta x = 0$, ſimulque differentiae ſequentium ordinum evaneſcunt. Quae cum ſint manifeſta a Poteſtate ſecunda incipiamus:

EXEMPLUM I.

Invenire differentias omnium ordinum poteſtatis x^2.

Cum hic ſit $y = x^2$, erit $y^I = (x + \omega)^2$, ideoque $\Delta y = 2\omega x + \omega\omega$; quae eſt differentia prima. Iam ob ω

 quan-

quantitatem conſtantem, erit $\Delta\Delta y = 2\,\omega\,\omega$, & $\Delta^3 y = 0$; $\Delta^4 y = 0$; &c.

EXEMPLUM II.

Invenire differentias omnium ordinum poteſtatis x^3.

Ponatur $y = x^3$; &, cum ſit $y^{\mathrm{I}} = (x + \omega)^3$, erit

$$\Delta y = 3\,\omega\,x\,x + 3\,\omega^2 x + \omega^3$$

quae eſt differentia prima. Deinde ob

$$\Delta .\, x\,x = 2\,\omega\,x + \omega\,\omega$$

erit

$$\Delta .\, 3\,\omega\,x\,x = 6\,\omega\,\omega\,x + 3\,\omega^3$$

&

$$\Delta .\, 3\,\omega^2 x = 3\,\omega^3; \quad \& \quad \Delta .\, \omega^3 = 0:$$

quibus collectis erit

$$\Delta\Delta y = 6\,\omega^2 x + 6\,\omega^3: \quad \text{atque} \quad \Delta^3 y = 6\,\omega^3:$$

Differentiae vero ſequentes evaneſcent.

EXEMPLUM III.

Invenire differentias omnium ordinum poteſtatis x^4.

Poſito $y = x^4$; ob $y^{\mathrm{I}} = (x + \omega)^4$ erit

$$\Delta y = 4\,\omega\,x^3 + 6\,\omega^2 x^2 + 4\,\omega^3 x + \omega^4;$$

quae eſt differentia prima. Tum ex praecedentibus eſt:

$$\Delta .\, 4\,\omega\,x^3 = 12\,\omega^2 x^2 + 12\,\omega^3 x + 4\,\omega^4$$
$$\Delta .\, 6\,\omega^2 x^2 = \ldots\ 12\,\omega^3 x + 6\,\omega^4$$
$$\Delta .\, 4\,\omega^3 x = \ldots\ldots + 4\,\omega^4$$
$$\Delta .\, \omega^4 = \ldots\ldots\ 0$$

His colligendis erit differentia ſecunda:

$$\Delta\Delta y = 12\,\omega^2 x^2 + 24\,\omega^3 x + 14\,\omega^4:$$

Quia deinde porro eſt:

$$\Delta .\, 12\,\omega^2 x^2 = 24\,\omega^3 x + 12\,\omega^4$$
$$\Delta .\, 24\,\omega^3 x = \ldots\ 24\,\omega^4$$
$$\Delta .\, 14\,\omega^4 = \ldots\ldots\ 0$$

pro-

prodibit differentia tertia:
$\Delta^3 y = 24\omega^3 x + 36\omega^4$
atque tandem differentia quarta:

$$\Delta^4 y = 24\omega^4$$

quae cum ſit conſtans, differentiae ſequentium ordinum evaneſcent.

EXEMPLUM IV.

Invenire differentias cuiuſvis ordinis poteſtatis x^n.

Ponatur $y = x^n$; &, cum ſit $y^{\mathrm{I}} = (x+\omega)^n$; $y^{\mathrm{II}} = (x+2\omega)^n$; $y^{\mathrm{III}} = (x+3\omega)^n$; &c. Poteſtates evolutae dabunt:

$$y = x^n$$

$$y^{\mathrm{I}} = x^n + \frac{n}{1}\omega x^{n-1} + \frac{n(n-1)}{1.2}\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1.2.3}\omega^3 x^{n-3} + \&c.$$

$$y^{\mathrm{II}} = x^n + \frac{n}{1}2\omega x^{n-1} + \frac{n(n-1)}{1.2}4\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1.2.3}8\omega^3 x^{n-3} + \&c.$$

$$y^{\mathrm{III}} = x^n + \frac{n}{1}3\omega x^{n-1} + \frac{n(n-1)}{1.2}9\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1.2.3}27\omega^3 x^{n-3} + \&c.$$

$$y^{\mathrm{IV}} = x^n + \frac{n}{1}4\omega x^{n-1} + \frac{n(n-1)}{1.2}16\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1.2.3}64\omega^3 x^{n-3} + \&c.$$

Hinc, differentiis ſumendis, prodibit:

$$\Delta y = \frac{n}{1}\omega x^{n-1} + \frac{n(n-1)}{1.2}\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1.2.3}\omega^3 x^{n-3} + \&c.$$

$$\Delta y^{\mathrm{I}} = \frac{n}{1}\omega x^{n-1} + \frac{n(n-1)}{1.2}3\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1.2.3}7\omega^3 x^{n-3} + \&c.$$

$$\Delta y^{II} = \frac{n}{1}\omega x^{n-1} + \frac{n(n-1)}{1 \cdot 2} 5\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} 19\omega^3 x^{n-3} + \&c.$$

$$\Delta y^{III} = \frac{n}{1}\omega x^{n-1} + \frac{n(n-1)}{1 \cdot 2} 7\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} 37\omega^3 x^{n-3} + \&c.$$

ſumantur denuo differentiae, atque obtinebitur:

$$\Delta\Delta y = n(n-1)\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} 6\omega^3 x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} 14\omega^4 x^{n-4} + \&c.$$

$$\Delta\Delta y^{I} = n(n-1)\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} 12\omega^3 x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} 50\omega^4 x^{n-4} + \&c.$$

$$\Delta\Delta y^{II} = n(n-1)\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} 18\omega^3 x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} 110\omega^4 x^{n-4} + \&c.$$

Ex his per ſubtractionem ulterius eruitur

$$\Delta^3 y = n(n-1)(n-2)\omega^3 x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} 36\omega^4 x^{n-4} + \&c.$$

$$\Delta^3 y^{I} = n(n-1)(n-2)\omega^3 x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} 60\omega^4 x^{n-4} + \&c.$$

atque porro

$$\Delta^4 y = n(n-1)(n-2)(n-3)\omega^4 x^{n-4} + \&c.$$

14. Quo lex, ſecundum quam iſtae differentiae poteſtatis

x^n

x^n progrediuntur, facilius perſpiciatur, ponamus primo brevitatis ergo:

$$A = \frac{n}{1}$$

$$B = \frac{n(n-1)}{1 \cdot 2}$$

$$C = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$$

$$D = \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}$$

$$E = \frac{n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

&c.

Deinde ſequens formetur Tabula, quae pro ſingulis differentiis inſerviet.

y	1;	0;	0;	0;	0;	0;	0;	0;	0;	&c.
Δy	0;	1;	1;	1;	1;	1;	1;	1;	1;	&c.
$\Delta^2 y$	0;	0;	2;	6;	14;	30;	62;	126;	254;	&c.
$\Delta^3 y$	0;	0;	0;	6;	36;	150;	540;	1806;	5796;	&c.
$\Delta^4 y$	0;	0;	0;	0;	24;	240;	1560;	8400;	40824;	&c.
$\Delta^5 y$	0;	0;	0;	0;	0;	120;	1800;	16800;	126000;	&c.
$\Delta^6 y$	0;	0;	0;	0;	0;	0;	720;	15120;	191520;	&c.
$\Delta^7 y$	0;	0;	0;	0;	0;	0;	0;	5040;	141120;	&c.

in qua Tabula numerus cuiuſvis ſeriei invenitur, ſi eiuſdem ſeriei praecedens ad numerum ſupra poſitum addatur, atque ſumma per indicem characteri Δ infixum multiplicetur. Sic, in ſerie differentiae $\Delta^5 y$ reſpondente, terminus 16800 invenitur, ſi praecedens 1800 ad ſupra ſcriptum 1560 addatur, atque ſumma 3360 per 5 multiplicetur.

15. Tabula ergo hac conſtituta, ſingulae differentiae Poteſtatis $x^n = y$ ſequenti modo ſe habebunt:

Δy

$$\Delta y = A\,\omega x^{n-1} + B\omega^2 x^{n-2} + C\omega^3 x^{n-3} + D\omega^4 x^{n-4} + \&c.$$

$$\Delta^2 y = \quad 2B\omega^2 x^{n-2} + 6C\omega^3 x^{n-3} + 14D\omega^4 x^{n-4} + \&c.$$

$$\Delta^3 y = \quad 6C\omega^3 x^{n-3} + 36D\omega^4 x^{n-4} + 150E\omega^5 x^{n-5} + \&c.$$

$$\Delta^4 y = \quad 24D\omega^4 x^{n-4} + 240E\omega^5 x^{n-5} + 1560F\omega^6 x^{n-6} + \&c.$$

Generatim autem poteſtatis x^n differentia ordinis m, ſeu $\Delta^m y$, ſequenti modo exprimetur.

Sit

$$I = \frac{n(n-1)(n-2)\ldots\ldots(n-m+1)}{1\cdot2\cdot3\ldots\ldots\ldots\ldots m};$$

$$K = \frac{n-m}{m+1}\,I;$$

$$L = \frac{n-m-1}{m+2}\,K;$$

$$M = \frac{n-m-2}{m+3}\,L;$$

&c.

Deinde vero ſit:

$$\alpha = m^m - \frac{m}{1}(m-1)^m + \frac{m(m-1)}{1\cdot2}(m-2)^m - \frac{m(m-1)(m-2)}{1\cdot2\cdot3}(m-3)^m + \&c.$$

$$\beta = m^{m+1} - \frac{m}{1}(m-1)^{m+1} + \frac{m(m-1)}{1\cdot2}(m-2)^{m+1} - \frac{m(m-1)(m-2)}{1\cdot2\cdot3}(m-3)^{m+1} + \&c.$$

$$\gamma = m^{m+2} - \frac{m}{1}(m-1)^{m+2} + \frac{m(m-1)}{1\cdot2}(m-2)^{m+2} - \frac{m(m-1)(m-2)}{1\cdot2\cdot3}(m-3)^{m+2} + \&c.$$

quibus valoribus inventis erit

$$\Delta^m y = \alpha I\omega^m x^{n-m} + \beta K\omega^{m+1}x^{n-m-1} + \gamma L\omega^{m+2}x^{n-m-2}\ \&c.$$

cu-

cuius expreſſionis ratio ex modo, quo ſingulae differentiae ex valoribus y, y', y'', y''', &c. eliciuntur, ſponte ſequitur.

16. Ex his perſpicuum eſt, ſi eſponens n fuerit numerus integer affirmativus, tandem ad differentias perveniri conſtantes, hisque ulteriores omnes eſſe $=0$. Sic erit

$$\Delta.\, x = \omega$$
$$\Delta^2.\, x^2 = 2\omega^2$$
$$\Delta^3.\, x^3 = 6\omega^3$$
$$\Delta^4.\, x^4 = 24\omega^4$$

& tandem

$$\Delta^n.\, x^n = 1.\ 2.\ 3.\ \ldots\ n.\ \omega^n$$

Omnis ergo functio rationalis integra tandem ad differentias conſtantes deducetur. Scilicet, functio ipſius x primi gradus, $ax+b$ differentiam primam iam habet conſtantem $= a\omega$. Functio ſecundi gradus $axx+bx+c$ differentiam ſecundam habebit conſtantem $= 2a\omega\omega$; functionis autem tertii gradus differentia tertia erit conſtans; quarti quarta, & ita porro.

17. Modus autem, quo invenimus differentias poteſtatis x^n, quoque latius patet, atque ad eas poteſtates, quarum exponens n eſt numerus negativus, vel fractus, vel adeo irrationalis, extenditur. Quod quo clarius appareat, differentias tantum primas praecipuarum huiuſmodi poteſtatum exhibebimus; quoniam lex differentiarum ſecundarum ac ſequentium non tam facile cernitur: erit ergo,

$$\Delta.\, x = \omega$$
$$\Delta.\, x^2 = 2\omega x + \omega^2$$
$$\Delta.\, x^3 = 3\omega x^2 + 3\omega^2 x + \omega^3$$
$$\Delta.\, x^4 = 4\omega x^3 + 6\omega^2 x^2 + 4\omega^3 x + \omega^4 \text{ \&c.}$$

Simili modo vero erit

Δ.

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$$\Delta . x^{-1} = -\frac{\omega}{x^2} + \frac{\omega^2}{x^3} - \frac{\omega^3}{x^4} + \ldots\ldots \&c.$$

$$\Delta . x^{-2} = -\frac{2\omega}{x^3} + \frac{3\omega^2}{x^4} - \frac{4\omega^3}{x^5} + \ldots\ldots \&c.$$

$$\Delta . x^{-3} = -\frac{3\omega}{x^4} + \frac{6\omega^2}{x^5} - \frac{10\omega^3}{x^6} + \ldots\ldots \&c.$$

$$\Delta . x^{-4} = -\frac{4\omega}{x^5} + \frac{10\omega^2}{x^6} - \frac{20\omega^3}{x^7} + \ldots\ldots \&c.$$

Et inde pro reliquis. Pariter erit

$$\Delta . x^{\frac{1}{2}} = \frac{\omega}{2x^{\frac{1}{2}}} - \frac{\omega^2}{8x^{\frac{3}{2}}} + \frac{\omega^3}{16x^{\frac{5}{2}}} - \ldots\ldots \&c.$$

$$\Delta . x^{\frac{1}{3}} = \frac{\omega}{3x^{\frac{2}{3}}} - \frac{\omega^2}{9x^{\frac{5}{3}}} + \frac{5\omega^3}{81x^{\frac{8}{3}}} - \ldots\ldots \&c.$$

$$\Delta . x^{-\frac{1}{2}} = -\frac{\omega}{2x^{\frac{3}{2}}} + \frac{3\omega^2}{8x^{\frac{5}{2}}} - \frac{5\omega^3}{16x^{\frac{7}{2}}} + \ldots \&c.$$

$$\Delta . x^{-\frac{1}{3}} = -\frac{\omega}{3x^{\frac{4}{3}}} + \frac{2\omega^2}{9x^{\frac{7}{3}}} - \frac{14\omega^3}{81x^{\frac{10}{3}}} + \ldots \&c.$$

18. Apparet itaque has differentias, ſi exponens ipſius x non fuerit numerus integer affirmativus, in infinitum progredi, ſeu ex terminorum numero infinito conſtare. Interim tamen eaedem differentiae quoque per expreſſionem finitam exhiberi poſſunt. Cum enim, poſito

$y = x^{-1} = \frac{1}{x}$, ſit $y^{\text{I}} = \frac{1}{x+\omega}$, erit $\Delta . x^{-1} = \Delta . \frac{1}{x} = \frac{1}{x+\omega} - \frac{1}{x}$; un-

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unde, ſi fractio $\frac{1}{x+\omega}$ in ſeriem convertatur, prodit expreſſio ſuperior. Simili modo erit

$$\Delta . x^{-2} = \Delta . \frac{1}{xx} = \frac{1}{(x+\omega)^2} - \frac{1}{xx},$$

atque pro irrationalibus erit

$$\Delta . \sqrt{x} = \sqrt{(x+\omega)} - \sqrt{x}, \ \& \ \Delta . \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{(x+\omega)}} - \frac{1}{\sqrt{x}};$$

quae formulae ſi more ſolito in ſeries explicentur, ſuperiores expreſſiones praebent.

19. Hoc vero modo quoque differentiae functionum, ſive fractarum ſive irrationalium, inveniri poſſunt: ſic ſi quaeratur differentia prima fractionis $\frac{1}{aa+xx}$ ponatur $y = \frac{1}{aa+xx}$; &, quia eſt $y^1 = \frac{1}{aa+xx+2\omega x+\omega^2}$ erit

$$\Delta y = \Delta . \frac{1}{aa+xx} = \frac{1}{aa+xx+2\omega x+\omega\omega} - \frac{1}{aa+xx},$$

quae expreſſio quoque in ſeriem infinitam converti poteſt. Ponatur $aa+xx = P$, & $2\omega x+\omega\omega = Q$; erit

$$\frac{1}{P+Q} = \frac{1}{P} - \frac{Q}{P^2} + \frac{Q^2}{P^3} - \frac{Q^3}{P^4} + \&c.$$

&

$$\Delta y = -\frac{Q}{P^2} + \frac{Q^2}{P^3} - \frac{Q^3}{P^4} + \&c.$$

Reſtitutis ergo loco P & Q valoribus erit:

$$\Delta y = \Delta . \frac{1}{aa+xx} = -\frac{2\omega x+\omega\omega}{(aa+xx)^2} + \frac{4\omega\omega xx+4\omega^3 x+\omega^4}{(aa+xx)^3}$$

$$- \frac{8\omega^3 x^3+12\omega^4 x^2+6\omega^5 x+\omega^6}{(aa+xx)^4} + \&c.$$

 qui

qui termini ſi ſecundum poteſtates ipſius ω ordinentur erit:

$$\Delta.\frac{1}{aa+xx}=-\frac{2\omega x}{(aa+xx)^2}+\frac{\omega^2(3xx-aa)}{(aa+xx)^3}-\frac{4\omega^3(x^3-aax)}{(aa+xx)^4}+\&c.$$

20. Similibus ſeriebus infinitis differentiae functionum irrationalium quoque exprimi poſſunt.

Sit propoſita iſta functio $y=\surd(aa+xx)$;
&, cum ſit $y^1=\surd(aa+xx+2\omega x+\omega\omega)$,
ponatur $aa+xx=P$, & $2\omega x+\omega\omega=Q$

$$\text{erit } \Delta y=\surd(P+Q)-\surd P=\frac{Q}{2\surd P}-\frac{QQ}{8P\surd P}+\frac{Q^3}{16PP\surd P}-\&c.$$

unde fiet

$$\Delta y=\Delta.\surd(aa+xx)=\frac{2\omega x+\omega\omega}{2\surd(aa+xx)}-\frac{4\omega^2x^2+4\omega^3x+\omega^4}{8(aa+xx)\surd(aa+xx)}+\&c.$$

vel

$$=\frac{\omega x}{\surd(aa+xx)}+\frac{aa\omega^2}{2(aa+xx)\surd(aa+xx)}-\frac{aa\omega^3x}{2(aa+xx)^2\surd(aa+xx)}\&c.$$

Hincque adeo colligimus functionis cuiuſcunque ipſius x, quae ſit y, differentiam hac forma exprimi poſſe, ut ſit

$$\Delta y= P\omega+Q\omega^2+R\omega^3+S\omega^4+\&c.$$

exiſtentibus P, Q, R, S, &c. certis ipſius x functionibus, quae quovis caſu ex functione y definiri poſſunt.

21. Neque etiam ex hac forma differentiae functionum tranſcendentium excluduntur, id quod ex ſequentibus exemplis clarius apparebit.

EXEMPLUM I.

Invenire differentiam primam logarithmi hyperbolici ipſius x.

Ponatur $y=lx$; & cum ſit $y^1=l(x+\omega)$, erit

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erit.

$$\Delta y = y^{1} - y = l(x+\omega) - l\,x = l\left(1+\frac{\omega}{x}\right).$$

Huiuſmodi autem logarithmum ſupra docuimus per ſeriem infinitam exprimere; qua adhibita, erit

$$\Delta y = \Delta . l\,x = \frac{\omega}{x} - \frac{\omega^{2}}{2xx} + \frac{\omega^{3}}{3x^{3}} - \frac{\omega^{4}}{4x^{4}} + \&c.$$

EXEMPLUM II.

Invenire differentiam primam quantitatis exponentialis a^{x}.

Poſito $y = a^{x}$ erit $y^{1} = a^{x+\omega} = a^{x} . a^{\omega}$:
at ſupra oſtendimus eſſe

$$a^{\omega} = 1 + \frac{\omega l a}{1} + \frac{\omega^{2}(la)^{2}}{1 . 2} + \frac{\omega^{3}(la)^{3}}{1 . 2 . 3} + \&c.$$

quo valore introducto erit

$$\Delta . a^{x} = y^{1} - y = \Delta y = \frac{a^{x}\omega l a}{1 .} + \frac{a^{x}\omega^{2}(l a)^{2}}{1 . 2} + \frac{a^{x}\omega^{3}(l a)^{3}}{1 . 2 . 3} + \&c.$$

EXEMPLUM III.

In circulo, cuius radius $= 1$, invenire differentiam ſinus arcus x.

Sit $\text{ſin}\, x = y$, erit $y^{1} = \text{ſin}(x+\omega)$,
unde $\Delta y = y^{1} - y = \text{ſin}(x+\omega) - \text{ſin}\, x$.
At eſt $\text{ſin}(x+\omega) = \text{coſ}\,\omega . \text{ſin}\, x + \text{ſin}\,\omega . \text{coſ}\, x$,
atque per ſeries infinitas oſtendimus eſſe,

$$\text{coſ}\,\omega = 1 - \frac{\omega^{2}}{1 . 2} + \frac{\omega^{4}}{1 . 2 . 3 . 4} - \frac{\omega^{6}}{1 . 2 . 3 . 4 . 5 . 6} + \&c.$$

&

$$\text{ſin}\,\omega = \omega - \frac{\omega^{3}}{1.2.3} + \frac{\omega^{5}}{1.2.3.4.5} - \frac{\omega^{7}}{1.2.3.4.5.6.7} + \&c.$$

quibus ſeriebus ſubſtitutis erit:

$$\Delta.\text{ſin}\,x = \omega.\text{coſ}\,x - \frac{\omega^2}{2}\,\text{ſin}\,x - \frac{\omega^3}{6}\,\text{coſ}\,x + \frac{\omega^4}{24}\,\text{ſin}\,x + \frac{\omega^5}{120}\,\text{coſ}\,x - \text{\&c.}$$

EXEMPLUM IV.

In circulo cuius radius $= 1$ *invenire differentiam coſinus arcus* x.

Poſito $y = \text{coſ}\,x$, ob $y^{\text{I}} = \text{coſ}(x+\omega)$
erit $y^{\text{I}} = \text{coſ}\,\omega.\,\text{coſ}\,x - \text{ſin}\,\omega.\,\text{ſin}\,x$
& $\Delta y = \text{coſ}\,\omega.\,\text{coſ}\,x - \text{ſin}\,\omega.\,\text{ſin}\,x - \text{coſ}\,x$
Seriebus ergo ante expoſitis adhibendis prodibit:

$$\Delta.\text{coſ}x = -\omega\text{ſin}x - \frac{\omega^2}{2}\,\text{coſ}x + \frac{\omega^3}{6}\,\text{ſin}x + \frac{\omega^4}{24}\,\text{coſ}x - \frac{\omega^5}{120}\,\text{ſin}x - \text{\&c.}$$

22. Cum igitur propoſita quacunque functione ipſius x, ſive algebraica ſive tranſcendente, quae ſit y, eius differentia prima eiuſmodi habeat formam ut ſit:

$$\Delta y = P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \text{\&c.}$$

ſi huius differentia denuo capiatur, patebit differentiam ſecundam ipſius y huiuſmodi formam eſſe habituram:

$$\Delta\Delta y = P\omega^2 + Q\omega^3 + R\omega^4 + \text{\&c.}$$

ſimilique modo differentia tertia ipſius y, erit huiuſmodi

$$\Delta^3 y = P\omega^3 + Q\omega^4 + R\omega^5 + \text{\&c.}$$

ſicque porro.

Ubi notandum eſt litteras P, Q, R, &c. hic non pro valoribus determinatis adhiberi, neque eadem littera in diverſis differentiis eandem functionem ipſius x denotari: ideo enim tantum iiſdem litteris utor, ne ſufficiens diverſarum litterarum numerus deficiat.

Ceterum *iſtae differentiarum formae* probe ſunt *notandae*, cum in Analyſi infinitorum maximum uſum offerant.

23.

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23. Cum igitur modum expoſuerim, quo cuiuſvis functionis differentia prima, ex eaque porro differentiae ſequentium ordinum inveniri queant; quippe quae ex valoribus functionis y ſucceſſivis y^{I}, y^{II}, y^{III}, y^{IV}, &c. reperiuntur: viciſſim ex differentiis ipſius y cuiuſque ordinis datis, iſti ipſi variati valores ipſius y elici poterunt. Erit enim

$$y^{\text{I}} = y + \Delta y$$
$$y^{\text{II}} = y + 2\Delta y + \Delta\Delta y$$
$$y^{\text{III}} = y + 3\Delta y + 3\Delta\Delta y + \Delta^3 y$$
$$y^{\text{IV}} = y + 4\Delta y + 6\Delta\Delta y + 4\Delta^3 y + \Delta^4 y$$

&c.

ubi coefficientes numerici iterum ex evolutione binomii naſcuntur. Quemadmodum ergo y^{I}, y^{II}, y^{III}, &c. ſunt valores ipſius y, qui oriuntur ſi loco x ſucceſſive ponantur hi valores $x+\omega$, $x+2\omega$, $x+3\omega$, &c. ſtatim valorem ipſius $y^{(n)}$ aſſignare poterimus, qui prodit ſi loco x ſcribatur $x+n\omega$, erit ſcilicet iſte valor:

$$y + \frac{n}{1}\Delta y + \frac{n(n-1)}{1.\,2}\Delta^2 y + \frac{n(n-1)\,(n-2)}{1.\,2.\,3}\Delta^3 y + \text{\&c.}$$

Hincque adeo etiam valores ipſius y praeberi poſſunt ſi n fuerit numerus negativus. Sic, ſi loco x ponatur $x-\omega$, functio y abibit in hanc formam:

$$y - \Delta y + \Delta^2 y - \Delta^3 y + \Delta^4 y - \text{\&c.}$$

ſin autem loco x ponatur $x-2\omega$, functio y tranſibit in:

$$y - 2\Delta y + 3\Delta^2 y - 4\Delta^3 y + 5\Delta^4 y - \text{\&c.}$$

24. Pauca quaedam addamus de methodo inverſa, qua, ſi detur differentia, ex ea ipſa illa functio, cuius eſt differentia, inveſtigari debeat. Cum autem hoc ſit difficillimum atque ſaepe numero ipſam analyſin infinitorum requirat, caſus

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ſus tantum quoſdam faciliores evolvamus. Primum igitur, regrediendo, ſi functionis cuiuſpiam differentiam invenerimus, viciſſim hac differentia propoſita, ipſa illa functio, unde eſt nata, exhiberi poterit. Sic, cum functionis $ax+b$ differentia ſit $a\omega$, ſi quaeratur cuiusnam functionis differentia ſit $a\omega$; reſponſio erit in promtu, eam functionem eſſe $ax+b$. In hac igitur reperitur quantitas conſtans b, quae in differentia non inerat, & quae propterea ab arbitrio noſtro pendet. Perpetuo autem ſi functionis cuiuſvis P differentia fuerit Q, quoque functionis P + A, (denotante A quantitatem quamcunque conſtantem,) differentia erit Q. Hinc, ſi iſta differentia Q proponatur, functio, ex qua ea eſt orta, erit P + A, atque idcirco determinatum valorem non habet, cum conſtans A ab arbitrio pendeat.

25. Vocemus eam functionem quaeſitam cuius differentia proponitur, SUMMAM; quod nomen commode adhibetur, cum quod ſumma differentiae opponi ſolet, tum etiam, quod functio quaeſita revera ſit ſumma omnium valorum praecedentium differentiae. Quemadmodum enim eſt $y^{I}=y+\Delta y$, & $y^{II}=y+\Delta y+\Delta y^{I}$, ſi valores ipſius y retro continuentur; ita, ut is, qui valori $x-\omega$ reſpondet, ſcribatur y_{I}, huncque praecedens y_{II}, & qui ultra praecedunt y_{III}, y_{IV}, y_{V}, &c. hincque ſeries formetur retrograda, cum ſuis differentiis:

$$y_{V}\,;\ y_{IV}\,;\ y_{III}\,;\ y_{II}\,;\ y_{I}\,;\ y$$

&

$$\Delta y_{V}\,;\ \Delta y_{IV}\,;\ \Delta y_{III}\,;\ \Delta y_{II}\,;\ \Delta y_{I}$$

erit

$$y=\Delta y_{I}+y_{I}$$

&

ob $y_{I}=\Delta y_{II}+y_{II}$, porroque $y_{II}=\Delta y_{III}+y_{III}$

erit utique

$$y=\Delta y_{I}+\Delta y_{II}+\Delta y_{III}+\Delta y_{IV}+\Delta y_{V}$$

&c.

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ſicque erit functio y, cuius differentia eſt Δy, ſumma omnium valorum antecedentium differentiae Δy, qui oriuntur, ſi loco x ſcribantur valores antecedentes $x-\omega$; $x-2\omega$; $x-3\omega$; &c.

26. Quemadmodum ad differentiam denotandam uſi ſumus ſigno Δ, ita ſummam indicabimus ſigno Σ: ſcilicet, ſi functionis y differentia fuerit z, erit $z=\Delta y$; unde, ſi y detur, differentiam z invenire ante docuimus. Quod ſi autem data ſit differentia z, eiuſque ſumma y reperiri debeat, fiet $y=\Sigma z$; atque adeo, ex aequatione $z=\Delta y$ regrediendo, formabitur haec aequatio $y=\Sigma z$; ubi conſtans quantitas quaecunque adiici poterit ob rationes ſupra datas; ex quo, aequatio $z=\Delta y$, ſi invertatur, dabit quoque $y=\Sigma z+C$. Deinde, cum quantitatis ay differentia ſit $a\Delta y=az$, erit $\Sigma\, az=ay$, ſi quidem a ſit quantitas conſtans. Quia ergo eſt $\Delta x=\omega$; erit $\Sigma\omega=x+C$ & $\Sigma a\omega=ax+C$; atque ob ω quantitatem conſtantem, erit $\Sigma\,\omega^2=\omega x+C$; $\Sigma\,\omega^3=\omega^2 x+C$; & ita porro.

27. Si igitur differentias poteſtatum ipſius x ſupra inventas invertamus, erit

$$\Sigma\omega = x\,;\quad \text{hincque}\quad \Sigma 1 = \frac{x}{\omega}\,;$$

Deinde habemus

$$\Sigma\,(2\omega x+\omega^2)=x^2\,;$$

unde fit

$$\Sigma x=\frac{x^2}{2\omega}-\Sigma\frac{\omega}{2}=\frac{x^2}{2\omega}-\frac{x}{2}\,.$$

Porro eſt

$$\Sigma\,(3\omega xx+3\omega^2 x+\omega^3)=x^3$$

ſeu

$$3\omega\Sigma x^2+3\omega^2\Sigma x+\omega^3\Sigma 1=x^3$$

ergo

$$\Sigma x^2=\frac{x^3}{3\omega}-\omega\Sigma x-\frac{\omega^2}{3}\Sigma 1$$

ſeu

$$\Sigma x^2=\frac{x^3}{3\omega}-\frac{x^2}{2}+\frac{\omega x}{6}$$

ſimi-

ſimili modo erit

$$\Sigma x^3 = \frac{x^4}{4\omega} - \frac{3\omega}{2}\Sigma x^2 - \omega^2 \Sigma x - \frac{\omega^3}{4}\Sigma 1$$

ubi, ſi loco Σx^2, Σx & $\Sigma 1$ valores ante inventi ſubſtituantur, reperietur:

$$\Sigma x^3 = \frac{x^4}{4\omega} - \frac{x^3}{2} + \frac{\omega xx}{4}.$$

Deinde, cum ſit

$$\Sigma x^4 = \frac{x^5}{5\omega} - 2\omega\Sigma x^3 - 2\omega^2 \Sigma x^2 - \omega^3 \Sigma x - \frac{\omega^4}{5}\Sigma 1$$

erit, adhibendis ſubſtitutionibus:

$$\Sigma x^4 = \frac{x^5}{5\omega} - \frac{1}{2}x^4 + \frac{1}{3}\omega x^3 - \frac{1}{30}\omega^3 x$$

ſimili modo ulterius progrediendo reperietur

$$\Sigma x^5 = \frac{x^6}{6\omega} - \frac{1}{2}x^5 + \frac{5}{12}\omega x^4 - \frac{1}{12}\omega^3 x^2$$

&

$$\Sigma x^6 = \frac{x^7}{7\omega} - \frac{1}{2}x^6 + \frac{1}{2}\omega x^5 - \frac{1}{6}\omega^3 x^3 + \frac{1}{42}\omega^5 x$$

quas expreſſiones infra facilius invenire docebimus.

28. Si ergo differentia propoſita fuerit functio rationalis integra ipſius x, eius ſumma, (ſeu ea functio, cuius ea eſt differentia) ex his formulis facile invenitur. Quia enim differentia ex aliquot poteſtatibus ipſius x conſtabit, quaeratur uniuſcuiuſque termini ſumma, omneſque iſtae ſummae colligantur.

EXEM-

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EXEMPLUM I.

Quaeratur functio, cuius differentia ſit $= axx + bx + c$.

Quaerantur ſingulorum terminorum ſummae ope formularum ante inventarum, erit

$$\Sigma axx = \frac{ax^3}{3\omega} - \frac{axx}{2} + \frac{a\omega x}{6}$$

& $$\Sigma bx = \ldots \frac{bxx}{2\omega} - \frac{bx}{2}$$

atque $$\Sigma c = \ldots\ldots \frac{cx}{\omega}$$

Hinc colligendo has ſummas erit

$$\Sigma(axx+bx+c) = \frac{a}{3\omega}x^3 - \frac{(a\omega-b)}{2\omega}x^2 + \frac{(a\omega^2-3b\omega+6c)}{6\omega}x + C$$

quae eſt functio quaeſita, cuius differentia eſt $axx+bx+c$.

EXEMPLUM II.

Quaeratur functio, cuius differentia eſt $x^4 - 2\omega^2 xx + \omega^4$.
Operationem ſimili modo inſtituendo habebitur.

$$\Sigma x^4 = \frac{1}{5\omega}x^5 - \frac{1}{2}x^4 + \frac{1}{3}\omega x^3 - \frac{1}{30}\omega^3 x$$

&

$$-\Sigma 2\omega^2 x^2 = \ldots -\frac{2\omega}{3}x^3 + \omega^2 x^2 - \frac{\omega^3}{3}x$$

atque

$$+\Sigma\omega^4 = \ldots\ldots\ldots\ldots + \omega^3 x$$

unde functio quaeſita erit:

$$\frac{1}{5\omega}x^5 - \frac{1}{2}x^4 - \frac{1}{3}\omega x^3 + \omega^2 x^2 + \frac{19}{30}\omega^3 x + C$$

Si enim hic loco x ponatur $x+\omega$, atque a quantitate reſultante ſubtrahatur iſta inventa, remanebit propoſita differentia $x^4 - 2\omega^2 x^2 + \omega^4$.

29. Si ſummas, quas pro poteſtatibus ipſius x invenimus, attentius inſpiciamus, in terminis primis, ſecundis, ac tertiis mox quidem legem obſervabimus, qua illi ſecundum ſingulas poteſtates progrediuntur: reliquorum autem terminorum lex non ita eſt perſpicua, ut ſummam poteſtatis x^n in genere inde colligere liceat. Interim tamen in ſequentibus docebitur eſſe:

$$\Sigma x^n =$$

$$\frac{x^{n+1}}{(n+1)\omega} - \frac{1}{2}x^n + \frac{1}{2}\cdot\frac{n\omega}{2.3}x^{n-1} - \frac{1}{6}\cdot\frac{n(n-1)(n-2)\omega^3}{2.\ 3.\ 4.\ 5}x^{n-3}$$

$$+\frac{1}{6}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)\omega^5}{2.\ 3.\ 4.\ 5.\ 6.\ 7}x^{n-5}$$

$$-\frac{3}{10}\cdot\frac{n(n-1)\ \ldots\ldots\ (n-6)\omega^7}{2.\ 3\ \ldots\ldots\ 8.\ 9}x^{n-7}$$

$$+\frac{5}{6}\cdot\frac{n(n-1)\ \ldots\ldots\ (n-8)\omega^9}{2.\ 3\ \ldots\ldots\ 10.\ 11}x^{n-9}$$

$$-\frac{691}{210}\cdot\frac{n(n-1)\ \ldots\ldots\ (n-10)\omega^{11}}{2.\ 3\ \ldots\ldots\ 12.\ 13}x^{n-11}$$

$$+\frac{35}{2}\cdot\frac{n(n-1)\ \ldots\ldots\ (n-12)\omega^{13}}{2.\ 3\ \ldots\ldots\ 14.\ 15}x^{n-13}$$

$$-\frac{3617}{30}\cdot\frac{n(n-1)\ \ldots\ldots\ (n-14)\omega^{15}}{2.\ 3\ \ldots\ldots\ 16.\ 17}x^{n-15}$$

$$+\frac{43867}{42}\cdot\frac{n(n-1)\ \ldots\ldots\ (n-16)\omega^{17}}{2.\ 3\ \ldots\ldots\ 18.\ 19}x^{n-17}$$

$$-\frac{1222277}{110}\cdot\frac{n(n-1)\ \ldots\ldots\ (n-18)\omega^{19}}{2.\ 3\ \ldots\ldots\ 20.\ 21}x^{n-19}$$

$$+$$

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$$+\frac{854513}{6}\cdot\frac{n(n-1)\;\ldots\ldots\;(n-20)\omega^{21}}{2\cdot 3\;\ldots\ldots\;22\cdot 23}x^{n-21}$$
$$-\frac{1181820455}{546}\cdot\frac{n(n-1)\;\ldots\ldots\;(n-22)\omega^{23}}{2\cdot 3\;\ldots\ldots\;24\cdot 25}x^{n-23}$$
$$+\frac{76977927}{2}\cdot\frac{n(n-1)\;\ldots\ldots\;(n-24)\omega^{25}}{2\cdot 3\;\ldots\ldots\;26\cdot 27}x^{n-25}$$
$$-\frac{23749461029}{30}\cdot\frac{n(n-1)\;\ldots\ldots\;(n-26)\omega^{27}}{2\cdot 3\;\ldots\ldots\;28\cdot 29}x^{n-27}$$
$$+\frac{8615841276005}{462}\cdot\frac{n(n-1)\;\ldots\ldots\;(n-28)\omega^{29}}{2\cdot 3\;\ldots\ldots\;30\cdot 31}x^{n-29}$$

&c. $+\,C$

cuius progreſſionis praecipuum momentum in coefficientibus mere numericis eſt ſitum, qui quemadmodum formentur, hic locus nondum eſt, ubi exponi queat.

30. Apparet autem niſi n ſit numerus integer affirmativus, hanc ſummae expreſſionem in infinitum progredi, neque hoc modo ſummam in forma finita exhiberi poſſe. Ceterum hic notandum eſt, non omnes poteſtates ipſius x propoſita x^n inferiores occurrere; deſunt enim termini x^{n-2}, x^{n-4}, x^{n-6}, x^{n-8}, &c. quippe quorum coefficientes ſunt $=0$, etiamſi termini ſecundi x^n coefficiens hanc legem non ſequatur, ſed ſit $=-\frac{1}{2}$. Poterunt ergo huius eſpreſſionis ope ſummae poteſtatum, quarum exponentes ſunt vel negativi vel fracti in forma infinita exhiberi ſolo excepto caſu quo $n=-1$, quia tum fit terminus $\frac{x^{n+1}}{(n+1)\omega}$ ob $n+1=0$ infinitus. Sic, poſito $n=-2$; erit

$$\Sigma \frac{1}{xx} = C - \frac{1}{\omega x} - \frac{1}{2xx} - \tfrac{1}{2} \cdot \frac{\omega}{3x^3} + \tfrac{1}{6} \cdot \frac{\omega^3}{5x^5}$$
$$- \tfrac{1}{6} \cdot \frac{\omega^5}{7x^7} + \tfrac{3}{10} \cdot \frac{\omega^7}{9x^9} - \tfrac{5}{6} \cdot \frac{\omega^9}{11x^{11}} + \frac{691}{210} \cdot \frac{\omega^{11}}{13x^{13}}$$
$$- \tfrac{35}{2} \cdot \frac{\omega^{13}}{15x^{15}} + \frac{3617}{30} \cdot \frac{\omega^{15}}{17x^{17}} - \&c.$$

31. Si ergo differentia propoſita fuerit poteſtas ipſius x quaecumque, eius ſumma hinc perpetuo aſſignari, ſeu functio, cuius ea ſit differentia, exhiberi poterit. Sin autem differentia propoſita aliam habeat formam, ut in poteſtates ipſius x, tanquam partes, diſtribui nequeat, tum ſumma difficillime ac ſaepenumero prorſus non inveniri poteſt: niſi forte pateat, eam ex quapiam functione eſſe ortam. Hanc ob cauſam conveniet plurium functionum differentias inveſtigare eaſque probe notare, ut ſi quando huiuſmodi differentia proponatur, eius ſumma, ſeu functio unde eſt orta, ſtatim exhiberi queat. Interim tamen methodus infinitorum plures regulas ſuppeditabit, quarum ope inventio ſummarum mirifice ſublevabitur.

32. Facilius autem ſaepe ex differentia propoſita reperitur ſumma quaeſita, ſi haec ex factoribus ſimplicibus conſtet, qui progreſſionem arithmeticam conſtituant, cuius differentia ſit ipſa quantitas ω. Sic, ſi propoſita fuerit functio $(x+\omega)(x+2\omega)$, eius differentia quaeratur: quia, poſito $x+\omega$ loco x, haec functio abit in $(x+2\omega)(x+3\omega)$, eius differentia erit $2\omega(x+2\omega)$. Quare viciſſim, ſi proponatur differentia $2\omega(x+2\omega)$, eius ſumma erit $(x+\omega)(x+2\omega)$, hinc ergo erit

$$\Sigma(x+2\omega) = \frac{1}{2\omega}(x+\omega)(x+2\omega).$$

Si-

Simili modo, ſi proponatur functio $(x+n\omega)(x+(n+1)\omega)$, cum ſit eius differentia $2\omega(x+(n+1)\omega)$ erit

$$\Sigma(x+(n+1)\omega)=\frac{1}{2\omega}(x+n\omega)(x+(n+1)\omega)$$

&

$$\Sigma(x+n\omega)=\frac{1}{2\omega}(x+(n-1)\omega)(x+n\omega).$$

33. Si functio ex pluribus factoribus conſtet, ut ſit $y=(x+(n-1)\omega)(x+n\omega)(x+(n+1)\omega)$, cum ſit

$$y^{1}=(x+n\omega)(x+(n+1)\omega)(x+(n+2)\omega),$$

erit

$$\Delta y=3\omega(x+n\omega)(x+(n+1)\omega)$$

ac propterea

$$\Sigma(x+n\omega)(x+(n+1)\omega)=\frac{1}{3\omega}(x+(n-1)\omega)(x+n\omega)(x+(n+1)\omega)$$

Pari modo reperietur eſſe:

$$\Sigma(x+n\omega)(x+(n+1)\omega)(x+(n+2)\omega)=$$
$$\frac{1}{4\omega}(x+(n-1)\omega)(x+n\omega)(x+(n+1)\omega)(x+(n+2)\omega).$$

unde lex inveniendi ſummas, ſi differentia ex pluribus huiuſmodi factoribus conſtet, ſponte patet. Quamvis autem hae differentiae ſint functiones rationales integrae, tamen earum ſummae hoc modo facilius reperiuntur, quam per methodum praecedentem.

34. Hinc quoque via patet ad differentiarum fractarum ſummas inveniendas. Sit enim propoſita fractio

$$y=\frac{1}{x+n\omega};\ \text{quia erit}\ y^{1}=\frac{1}{x+(n+1)\omega}$$

erit

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erit

$$\Delta y = \frac{1}{x+(n+1)\omega} - \frac{1}{x+n\omega} = \frac{-\omega}{(x+n\omega)(x+(n+1)\omega)}$$

ac propterea

$$\Sigma \frac{1}{(x+n\omega)\ (x+(n+1)\omega} = -\frac{1}{\omega} \cdot \frac{1}{x+n\omega}$$

Sit porro

$$y = \frac{1}{(x+n\omega)\ (x+(n+1)\omega)}$$

ob $y^{I} = \frac{1}{(x+(n+1)\omega)\ (x+(n+2)\omega)}$

erit

$$\Delta y = \frac{-2\omega}{(x+n\omega)\ (x+(n+1)\omega)\ (x+(n+2)\omega)}$$

Hinc ideo fiet

$$\Sigma \frac{1}{(x+n\omega)\ (x+(n+1)\omega)\ (x+(n+2)\omega)}$$
$$= \frac{-1}{2\omega} \cdot \frac{1}{(x+n\omega)\ (x+(n+1)\omega)}$$

Simili modo erit porro

$$\Sigma \frac{1}{(x+n\omega)\ (x+(n+1)\omega)\ (x+(n+2)\omega)\ (x+(n+3)\omega)}$$
$$= \frac{-1}{3\omega} \cdot \frac{1}{(x+n\omega)\ (x+(n+1)\omega)\ (x+(n+2)\omega)}.$$

35. Modus iſte ſummandi probe eſt tenendus, quia huiuſmodi differentiarum ſummae per praecedentem methodum inveniri non poſſunt. Quodſi autem differentia inſuper ha-

habeat numeratorem, vel factores denominatoris non in arithmetica progreſſione procedant, tum tutiſſimus modus inveſtigandi ſummas eſt, ut differentia propoſita in ſuas fractiones ſimplices reſolvatur, quarum ſingulae etſi ſummari nequeunt, tamen binis coniungendis toties ſumma inveniri poteſt, quoties id quidem fieri licet, tantum enim erit diſpiciendum, utrum ſumma ope huius formulae inveniri queat:

$$\Sigma\frac{1}{x+(n+1)\omega}-\Sigma\frac{1}{x+n\omega}=\frac{1}{x+n\omega}$$

etſi enim neutra harum ſummarum per ſe exhiberi poteſt, tamen earum differentia cognoſcitur.

36. His igitur caſibus negotium redit ad reſolutionem cuiuſque fractionis in fractiones ſuas ſimplices, quae in ſuperiori libro fuſius eſt oſtenſa. Quemadmodum ergo eius beneficio ſummae inveniri queant, aliquot exemplis docebimus.

EXEMPLUM I.

Quaeratur ſumma, cuius differentia ſit

$$\frac{3x+2\omega}{x(x+\omega)(x+2\omega)}$$

Reſolvatur haec differentia propoſita in ſuas fractiones ſimplices, quae erunt

$$\frac{1}{\omega}\cdot\frac{1}{x}+\frac{1}{\omega}\cdot\frac{1}{x+\omega}-\frac{2}{\omega}\cdot\frac{1}{x+2\omega}.$$

Cum iam ſit ex ſuperiori formula:

$$\Sigma\frac{1}{x+n\omega}=\Sigma\frac{1}{x+(n+1)\omega}-\frac{1}{x+n\omega}$$

erit $\Sigma\frac{1}{x}=\Sigma\frac{1}{x+\omega}-\frac{1}{x}$

Hinc erit ſumma quaeſita

$$\frac{1}{\omega}\Sigma\frac{1}{x}+\frac{1}{\omega}\Sigma\frac{1}{x+\omega}-\frac{2}{\omega}\Sigma\frac{1}{x+2\omega}=$$

$$\frac{2}{\omega}\Sigma\frac{1}{x+\omega}-\frac{2}{\omega}\Sigma\frac{1}{x+2\omega}-\frac{1}{\omega x}$$

at eſt

$$\Sigma\frac{1}{x+\omega}=\Sigma\frac{1}{x+2\omega}-\frac{1}{x+\omega};$$

unde ſumma quaeſita erit

$$-\frac{1}{\omega x}-\frac{2}{\omega(x+\omega)}=\frac{-3x-\omega}{\omega x(x+\omega)}.$$

EXEMPLUM II.

Quaeratur ſumma, cuius differentia eſt $\frac{3\omega}{x(x+3\omega)}$.

Poſita hac differentia $=z$, erit $z=\frac{1}{x}-\frac{1}{x+3\omega}$ ideoque

$$\Sigma z=\Sigma\frac{1}{x}-\Sigma\frac{1}{x+3\omega}=\Sigma\frac{1}{x+\omega}-$$

$$\Sigma\frac{1}{x+3\omega}-\frac{1}{x}=\Sigma\frac{1}{x+2\omega}-\Sigma\frac{1}{x+3\omega}-\frac{1}{x}-\frac{1}{x+\omega}$$

$$=-\frac{1}{x}-\frac{1}{x+\omega}-\frac{1}{x+2\omega}.$$

quae eſt ſumma quaeſita. Quoties ergo hoc modo ſigna ſummatoria Σ ſeſe tandem tollunt, toties differentiae propoſitae ſumma exhiberi poterit; ſin autem haec deſtructio non ſuccedat, ſignum hoc eſt, ſummam inveniri non poſſe.

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CAPUT II.
DE USU DIFFERENTIARUM IN DOCTRINA SERIERUM.

37.

Naturam ſerierum per differentias maxime illuſtrari, ex primis rudimentis ſatis eſt notum. Progreſſionis enim arithmeticae, quae primum conſiderari ſolet, praecipua proprietas in hoc verſatur, ut eius differentiae primae ſint inter ſe aequales; hinc differentiae ſecundae ac reliquae omnes erunt cyphrae. Dantur deinde ſeries, quarum differentiae ſecundae demum ſunt aequales, quae hanc ob rem *ſecundi ordinis* commode appellantur, dum progreſſiones arithmeticae ſeries *primi ordinis* vocantur. Porro igitur ſeries *tertii ordinis* erunt, quarum differentiae tertiae ſunt conſtantes, atque ad *quartum ordinem* & ſequentes eae referentur ſeries, quarum differentiae quartae, & ulteriores demum ſunt conſtantes.

38. In hac diviſione infinita ſerierum genera comprehenduntur, neque tamen omnes ſeries ad haec genera revocare licet. Occurrunt enim innumerabiles ſeries, quae, differentiis ſumendis, nunquam ad terminos conſtantes deducunt: cuiuſmodi, praeter innumeras alias ſunt progreſſiones geometricae, quae nunquam praebent differentias conſtantes, uti ex hoc exemplo videre licet.

1, 2, 4, 8, 16, 32, 64, 128, &c.
1, 2, 4, 8, 16, 32, 64, &c.
1, 2, 4, 8, 16, 32, &c.

Cum enim ſeries differentiarum cuiuſque ordinis aequalis ſit ipſi ſeriei propoſitae, aequalitas differentiarum prorſus excluditur. Quocirca plures ſerierum claſſes conſtitui debebunt,

 qua-

quarum una tantum in hos ordines, qui tandem ad differentias conſtantes revocantur, ſubdividitur; quam claſſem in hoc capite potiſſimum conſiderabimus.

39. Duae autem res ad naturam ſerierum cognoſcendam imprimis requiri ſolent, Terminus generalis atque Summa ſeu Terminus ſummatorius. Terminus generalis eſt expreſſio indefinita, quae unumquemque ſeriei terminum complectitur, atque eiuſmodi propterea eſt functio quantitatis variabilis x, quae, poſito $x = 1$, terminum ſeriei primum exhibet; ſecundum vero poſito $x = 2$; tertium poſito $x = 3$; quartum poſito $x = 4$; & ita porro. Cognito ergo termino generali, quotuſcumque ſeriei terminus invenietur, etiamſi lex, qua ſinguli termini cohaerent, non reſpiciatur. Sic verbi gratia ponendo $x = 1000$, ſtatim terminus milleſimus cognoſcetur. Ita huius ſeriei

$$1,\ 6,\ 15,\ 28,\ 45,\ 66,\ 91,\ 120,\ \&c.$$

Terminus generalis eſt $2xx - x$; poſito enim $x = 1$, haec formula dat terminum primum 1; poſito $x = 2$, oritur terminus ſecundus 6; ſi ponatur $x = 3$, oritur tertius 15; &c. unde patet huius ſeriei terminum centeſimum, poſito $x = 100$ fore $= 2.\ 10000 - 100 = 19900$.

40. Indices ſeu exponentes in qualibet ſerie vocantur numeri, qui indicant quotus quiſque terminus ſit in ordine: ſic, termini primi index erit 1, ſecundi 2, tertii 3, & ita porro. Hinc indices ſingulis cuiuſque ſeriei terminis inſcribi ſolent, hoc modo

INDICES.

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \&c.$$

TERMINI.

$$A,\ B,\ C,\ D,\ E,\ F,\ G,\ \&c.$$

unde ſtatim patet G eſſe ſeriei propoſitae terminum ſeptimum, cum eius index ſit 7. Hinc terminus generalis nil aliud erit, niſi terminus ſeriei, cuius index vel exponens eſt nu-

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numerus indefinitus x. Quemadmodum ergo in quolibet ſerierum ordine, quarum differentiae vel primae, vel ſecundae, vel aliae ſequentes ſunt conſtantes, terminum generalem inveniri oporteat, primum docebimus: tum vero ad inveſtigationem ſummae ſummus progreſſuri.

41. Incipiamus ab ordine primo, qui continet progreſſiones arithmeticas, quarum differentiae primae ſunt conſtantes; ſitque a terminus ſeriei primus, & b terminus primus ſeriei differentiarum, cui ſequentes omnes ſunt aequales: unde ſeries ita erit comparata.

INDICES.

$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6,$

TERMINI.

$a, \quad a+b, \quad a+2b, \quad a+3b, \quad a+4b, \quad a+5b,$ &c.

DIFFERENTIAE.

$b, \quad b, \quad b, \quad b, \quad b,$ &c.

Ex qua ſtatim patet, terminum, cuius index ſit $=x$, fore $a+(x-1)b$, eritque ergo terminus generalis $=bx+a-b$, qui ex terminis primis cum ipſius ſeriei, tum ſeriei differentiarum componitur. Quodſi autem terminus ſecundus ſeriei $a+b$ vocetur a^{I}, ob $b=a^{\text{I}}-a$, erit terminus generalis $=(a^{\text{I}}-a)x+2a-a^{\text{I}}=a^{\text{I}}(x-1)-a(x-2)$ unde ex cognitis terminis primo & ſecundo progreſſionis arithmeticae, eius terminus generalis formabitur.

42. Sint in ſerie ſecundi ordinis termini primi; ipſius ſeriei $=a$; differentiarum primarum $=b$; differentiarum ſecundarum $=c$; eritque ipſa ſeries cum ſuis differentiis ita comparata.

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			INDICES.				
1,	2,	3,	4,	5,	6,	7,	
			TERMINI.				
a;	$a+b$;	$a+2b+c$;	$a+3b+3c$;	$a+4b+6c$;	$a+5b+10c$;	$a+6b+15c$;	&c.
			DIFFER. I.				
b;	$b+c$;	$b+2c$;	$b+3c$;	$b+4c$;	$b+5c$;	&c.	
			DIFFER. II.				
c,	c,	c,	c,	c,	&c.		

ex cuius inſpectione liquet terminum, cuius index $= x$ fore $= a + (x-1)b + \frac{(x-1)}{1.}\frac{(x-2)}{2.}c$; qui ergo eſt terminus generalis ſeriei propoſitae. Ponatur autem ipſius ſeriei terminus ſecundus $= a^{\text{I}}$, terminus tertius $= a^{\text{II}}$, cum ſit $b = a^{\text{I}} - a$; & $c = a^{\text{II}} - 2a^{\text{I}} + a$; uti ex natura differentiarum (§. 10.) intelligitur, erit terminus generalis

$$a + (x-1)(a^{\text{I}} - a) + \frac{(x-1)(x-2)}{1.\quad 2}(a^{\text{II}} - 2a^{\text{I}} + a)$$

qui reducitur ad hanc formam

$$\frac{a^{\text{II}}(x-1)(x-2)}{1.\quad 2} - \frac{2a^{\text{I}}(x-1)(x-3)}{1.\quad 2.} + \frac{a(x-2)(x-3)}{1.\quad 2}$$

vel etiam ad hanc

$$\frac{a^{\text{II}}}{2}(x-1)(x-2) - \frac{2a^{\text{I}}}{2}(x-1)(x-3) + \frac{a}{2}(x-2)(x-3)$$

aut denique ad hanc

$$\tfrac{1}{2}(x-1)(x-2)(x-3)\left(\frac{a^{\text{II}}}{x-3} - \frac{2a^{\text{I}}}{x-2} + \frac{a}{x-1}\right);$$

ideoque ex tribus terminis ipſius ſeriei definitur.

43. Sit ſeries tertii ordinis a, a^{I}, a^{II}, a^{III}, a^{IV}, &c. eius differentiae primae b, b^{I}, b^{II}, b^{III}, &c. & differentiae

tiae ſecundae c, c^{I}, c^{II}, c^{III}, &c. & tertiae d, d, d, &c. quippe quae ſunt conſtantes.

Indices	1,	2,	3,	4,	5,	6,	
Termini	a,	a^{I},	a^{II},	a^{III},	a^{IV},	a^{V},	&c.
Differ. I.		b,	b^{I},	b^{II},	b^{III},	b^{IV},	&c.
Differ. II.			c,	c^{I},	c^{II},	c^{III},	&c.
Differ. III.				d,	d,	d,	&c.

Quia eſt $a^{I} = a + b$; $a^{II} = a + 2b + c$, $a^{III} = a + 3b + 3c + d$; $a^{IV} = a + 4b + 6c + 4d$; &c. erit terminus generalis, ſeu is cuius index eſt x,

$$a + \frac{(x-1)}{1}b + \frac{(x-1)(x-2)}{1 \cdot 2}c + \frac{(x-1)(x-2)(x-3)}{1 \cdot 2 \cdot 3}d$$

Sicque terminus generalis ex differentiis formabitur.

Cum autem porro ſit

$$b = a^{I} - a;\ c = a^{II} - 2a^{I} + a;\ d = a^{III} - 3a^{II} + 3a^{I} - a$$

ſi hi valores ſubſtituantur erit terminus generalis

$$a^{III}\frac{(x-1)(x-2)(x-3)}{1 \cdot 2 \cdot 3} - 3a^{II}\frac{(x-1)(x-2(x-4)}{1 \cdot 2 \cdot 3}$$

$$+ 3a^{I}\frac{(x-1)(x-3)(x-4)}{1 \cdot 2 \cdot 3} - a\frac{(x-2)(x-3)(x-4)}{1 \cdot 2 \cdot 3}$$

qui etiam hoc modo exprimetur, ut ſit

$$= \frac{(x-1)(x-2)(x-3)(x-4)}{1 \cdot 2 \cdot 3}\left(\frac{a^{III}}{x-4} - \frac{3a^{II}}{x-3} + \frac{3a^{I}}{x-2} - \frac{a}{x-1}\right).$$

44. Sit nunc ſeries cuiuſcunque ordinis propoſita:

Indices	1,	2,	3,	4,	5,	6,	
Termini	a,	a^{I},	a^{II},	a^{III},	a^{IV},	a^{V},	&c.
Differ. I.		b,	b^{I},	b^{II},	b^{III},	b^{IV},	&c.
Differ. II.			c,	c^{I},	c^{II},	c^{III},	&c.

Dif-

Differ. III. d, d^{I}, d^{II}, &c.

Differ. IV. e, e^{I}, &c.

Differ. V. f, &c.

ex ipſius ſeriei termino primo, atque ex differentiarum terminis primis b, c, d, e, f, &c. terminus generalis ita exprimetur ut ſit:

$$a + \frac{(x-1)}{1} b + \frac{(x-1)(x-2)}{1 \cdot 2} c + \frac{(x-1)(x-2)(x-3)}{1 \cdot 2 \cdot 3} d + \frac{(x-1)(x-2)(x-3)(x-4)}{1 \cdot 2 \cdot 3 \cdot 4} e + \text{\&c.}$$

donec ad differentias conſtantes perveniatur. Ex quo patet, ſi nunquam prodeant differentiae conſtantes, terminum generalem per expreſſionem infinitam exhiberi.

45. Quia differentiae ex ipſis terminis ſeriei formantur, ſi earum valores ſubſtituantur, prodibit terminus generalis in eiuſmodi forma expreſſus, cuiuſmodi pro ſeriebus primi, ſecundi, & tertii ordinis exhibuimus. Scilicet, pro ſeriebus ordinis quarti, erit terminus generalis

$$\frac{(x-1)(x-2)(x-3)(x-4)(x-5)}{1 \cdot 2 \cdot 3 \cdot 4} \times \left(\frac{a^{IV}}{x-5} - \frac{4a^{III}}{x-4} + \frac{6a^{II}}{x-3} - \frac{4a^{I}}{x-2} + \frac{a}{x-1}\right)$$

unde lex, qua ſequentium ordinum termini generales componuntur, facile perſpicitur. Ex his autem patet pro quovis ordine terminum generalem fore functionem ipſius x rationalem integram, in qua maxima ipſius x dimenſio congruat cum ordine, ad quem ſeries refertur. Ita ſerierum primi ordinis erit terminus generalis functio primi gradus, ſecundi ordinis ſecundi gradus, & ita porro.

46.

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46. Differentiae autem, uti ſupra vidimus, ex ipſis terminis ſeriei ita reſultant, ut ſit

$$\begin{aligned} b &= a^{I} - a \\ b^{I} &= a^{II} - a^{I} \\ b^{II} &= a^{III} - a^{II} \\ &\&c. \\ c &= a^{II} - 2a^{I} + a \\ c^{I} &= a^{III} - 2a^{II} + a^{I} \\ c^{II} &= a^{IV} - 2a^{III} + a^{II} \\ &\&c. \\ d &= a^{III} - 3a^{II} + 3a^{I} - a \\ d^{I} &= a^{IV} - 3a^{III} + 3a^{II} - a^{I} \\ d^{II} &= a^{V} - 3a^{IV} + 3a^{III} - a^{II} \\ &\&c. \end{aligned}$$

Quare cum in ſeriebus primi ordinis ſint omnes valores ipſius $c = 0$; erit
$a^{II} = 2a^{I} - a$; $a^{III} = 2a^{II} - a^{I}$; $a^{IV} = 2a^{III} - a^{II}$; &c.
unde patet has ſeries ſimul eſſe recurrentes, & ſcalam relationis eſſe $2, -1$. Deinde, cum in ſeriebus ſecundi ordinis ſint omnes valores ipſius $d = 0$, erit
$a^{III} = 3a^{II} - 3a^{I} + a$; $a^{IV} = 3a^{III} - 3a^{II} + a^{I}$; &c.
ideoque & hae erunt recurrentes ſcala relationis exiſtente

$$3, -3, +1.$$

Simili modo apparebit omnes huius claſſis ſeries, cuiuſcunque ſint ordinis, ſimul ad claſſem ſerierum recurrentium pertinere, atque ita quidem, ut ſcala relationis conſtet ex coefficientibus poteſtatis binomii, uno gradu ſuperioris, quam eſt ordo, ad quem ſeries refertur.

47. Quia vero pro ſeriebus primi ordinis quoque omnes valores ipſius d & e, & ſequentium differentiarum omnium ſunt $= 0$, erit quoque in his

$$\begin{aligned} a^{III} &= 3a^{II} - 3a^{I} + a \\ a^{IV} &= 3a^{III} - 3a^{II} + a^{I} \\ &\&c. \end{aligned}$$

aut

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aut
$$a^{IV} = 4a^{III} - 6a^{II} + 4a^{I} - a$$
$$a^{V} = 4a^{IV} - 6a^{III} + 4a^{II} - a^{I}$$
&c.

Pertinebunt ergo & hinc ad ſeries recurrentes idque infinitis modis, cum ſcalae relationis eſſe queant:

$3, -3, +1;\ 4, -6, +4, -1;\ 5, -10, +10, -5, +1;$
&c.

Similique modo intelligitur unamquamque ſeriem huius, quam tractamus, claſſis ſimul eſſe ſeriem recurrentem innumeris modis: ſcala enim relationis erit

$$\frac{n}{1}, -\frac{n(n-1)}{1 \cdot 2}, +\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}, -\frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4},$$

dummodo n ſit numerus integer maior, quam numerus quo ordo indicatur. Orietur ergo haec ſeries quoque ex evolutione fractionis, cuius denominator eſt $(1-y)^n$, prouti in ſuperiori libro de ſeriebus recurrentibus fuſius eſt oſtenſum.

48. Quemadmodum vidimus, omnium huius claſſis ſerierum, cuiuſcunque ſint ordinis, terminos generales eſſe functiones ipſius x rationales integras, ita viciſſim apparebit omnes ſeries, quarum termini generales ſint huiuſmodi functiones ipſius x, ad hanc claſſem pertinere, atque tandem ad differentias conſtantes perduci. Et quidem, ſi terminus generalis fuerit functio primi gradus $ax+b$, dum ſeries inde orta erit primi ordinis ſeu arithmetica, differentias primas habebit conſtantes. Sin autem terminus generalis fuerit functio ſecundi gradus in hac forma $axx+bx+c$ contenta, tum ſeries ex eo oriunda, dum loco x ſucceſſive numeri 1, 2, 3, 4, 5, &c. ſubſtituuntur, erit ordinis ſecundi, atque differentias ſecundas habebit conſtantes: ſimili modo, terminus generalis tertii gradus ax^3+bx^2+cx+d dabit ſeriem tertii ordinis atque ita porro.

49. Ex termino enim generali non ſolum omnes ſeriei termini inveniuntur, ſed etiam ſeries differentiarum tam prima-

marum quam ſequentium deduci poſſunt. Cum enim, ſi ſeriei terminus primus ſubtrahatur a ſecundo, prodeat ſeriei differentiarum terminus primus: ſecundus autem, ſi ipſius ſeriei terminus ſecundus a tertio auferatur, ita ſeriei differentiarum obtinebitur terminus, cuius index eſt x, ſi ipſius ſeriei terminus, cuius index eſt x, ſubtrahatur a ſequente cuius index eſt $x+1$. Quare ſi in termino ſeriei generali loco x ponatur $x+1$, ab hocque valore terminus generalis ſubtrahatur, remanebit terminus generalis ſeriei differentiarum: ſi igitur X fuerit ſeriei terminus generalis, erit eius differentia ΔX, (quae modo in praecedente capite oſtenſo invenietur, ſi ſtatuatur ibi $\omega = 1$) terminus generalis ſeriei differentiarum primarum. Simili igitur modo erit $\Delta\Delta X$ terminus generalis ſeriei differentiarum ſecundarum; $\Delta^3 X$ tertiarum, ſicque deinceps.

50. Quodſi autem terminus generalis X fuerit functio rationalis integra, in qua maximus exponens poteſtatis ipſius x ſit n, ex capite praecedente colligitur, eius differentiam ΔX fore functionem uno gradu inferiorem, nempe gradus $n-1$. Hincque porro $\Delta\Delta X$ erit functio gradus $n-2$, & $\Delta^3 X$ functio gradus $n-3$, & ita porro. Quare, ſi X fuerit functio primi gradus, uti $ax+b$, tum eius differentia ΔX erit conſtans $= a$; quae cum ſit terminus generalis ſeriei primarum differentiarum, perſpicitur ſeriem, cuius terminus generalis X ſit functio primi gradus, fore arithmeticam ſeu primi ordinis. Simili modo ſi terminus generalis X fuerit functio ſecundi gradus ob $\Delta\Delta X$ conſtantem, ſeries inde orta differentias ſecundas habebit conſtantes, eritque propterea ordinis ſecundi; ſicque perpetuo, cuius gradus fuerit functio X terminum generalem conſtituens, eiuſdem ordinis erit ſeries ex eo nata.

51. Hanc ob rem ſeries poteſtatum numerorum naturalium ad differentias conſtantes perveniunt, uti ex ſequenti ſchemate ſit manifeſtum.

POTEST. I.

	1,	2,	3,	4,	5,	6,	7,	8,	&c.
Differ. I.	1,	1,	1,	1,	1,	1,	1,	&c.	

POTEST. II.

	1,	4,	9,	16,	25,	36,	49,	64,	&c.
Differ. I.	3,	5,	7,	9,	11,	13,	15,	&c.	
Differ. II.	2,	2,	2,	2,	2,	2,	&c.		

POTEST. III.

	1,	8,	27,	64,	125,	216,	343,	&c.
Differ. I.	7,	19,	37,	61,	91,	127,	&c.	
Differ. II.	12,	18,	24,	30,	36,	&c.		
Differ. III.	6,	6,	6,	6,	&c.			

POTEST. IV.

	1,	16,	81,	256,	625,	1296,	2401,	&c.
Differ. I.	15,	65,	175,	369,	671,	1105,	&c.	
Differ. II.	50,	110,	194,	302,	434,	&c.		
Differ. III.	60,	84,	108,	132,	&c.			
Differ. IV.	24,	24,	24,	&c.				

Quae igitur in capite praecedente de differentiis cuiuſque ordinis inveniendis ſunt praecepta, ea hic inſervient ad terminos generales differentiarum quarumvis, quae ex ſeriebus naſcuntur, inveniendos.

52. Si terminus generalis cuiuſquam ſeriei fuerit cognitus, eius ope non ſolum omnes eius termini in infinitum inveniri, ſed etiam ſeries retro continuari, eiuſque termini, quorum exponentes ſint numeri negativi, exhiberi poterunt, loco x numeros negativos ſubſtituendo: ſic, ſi terminus generalis fuerit $\frac{xx+3x}{2}$ ponendo loco x tam negativos quam affirmativos indices, ſeries utrinque continuata erit huiuſmodi.

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INDICES.

&c.$-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6,$ &c.

SERIES.

&c.$+5, +2, 0, -1, -1, 0, 2, 5, 9, 14, 20, 27,$ &c.

Differ. I. $-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7,$ &c.

Differ. II. $1, 1, 1, 1, 1, 1, 1, 1, 1, 1,$ &c.

Cum igitur ex differentiis terminus generalis formetur, quaeque ſeries ex differentiis retro continuari poterit; ita quidem, ut, ſi differentiae tandem fiant conſtantes, hi termini finite exhiberi, contra vero per expreſſionem infinitam aſſignari queant: Quin etiam ex termino generali, ii termini, quorum indices ſunt fracti, definientur, in quo ſerierum INTERPOLATIO continetur.

53. His de termino ſerierum generali monitis, progrediamur ad ſummam, ſeu terminum ſummatorium ſerierum cuiuſque ordinis inveſtigandum. Propoſita autem quacumque ſerie, TERMINUS *ſummatorius* eſt functio ipſius x, quae aequalis eſt ſummae tot terminorum ſeriei, quot unitates continet numerus x. Ita ergo terminus ſummatorius erit comparatus, ut ſi ponatur $x = 1$, prodeat terminus primus ſeriei; ſin autem ponatur $x = 2$, ut prodeat ſumma primi & ſecundi, facto autem $x = 3$, ſumma primi, ſecundi ac tertii; ſicque deinceps. Hinc, ſi ex ſerie propoſita nova ſeries formetur, cuius primus terminus aequalis ſit primo illius, ſecundus aequalis ſummae duorum, tertius aequalis ſummae trium, atque ita porro, haec nova ſeries vocatur illius *ſummatrix*, huiuſque ſeriei ſummatricis terminus generalis erit terminus ſummatorius ſeriei propoſitae: ex quo inventio termini ſummatorii ad iuventionem termini generalis revocatur.

54. Sit ergo ſeries propoſita haec

$$a, \; a^{I}, \; a^{II}, \; a^{III}, \; a^{IV}, \; a^{V}, \; \&c.$$

huiuſque ſeriei ſummatrix ſit

$$A,\ A^{I},\ A^{II},\ A^{III},\ A^{IV},\ A^{V},\ \&c.$$

erit ex eius natura modo expoſita:

$$A = a$$
$$A^{I} = a + a^{I}$$
$$A^{II} = a + a^{I} + a^{II}$$
$$A^{III} = a + a^{I} + a^{II} + a^{III}$$
$$A^{IV} = a + a^{I} + a^{II} + a^{III} + a^{IV}$$
&c.

Hinc ſeriei ſummatricis differentiae erunt:

$$A^{I} - A = a^{I};\ A^{II} - A^{I} = a^{II};\ A^{III} - A^{II} = a^{III};\ \&c.$$

unde ſeries propoſita termino primo minuta erit ſeries differentiarum primarum ſeriei ſummatricis. Quod ſi igitur ſeriei ſummatrici praefigatur terminus $= 0$, ut habeatur:

$$0,\ A,\ A^{I},\ A^{II},\ A^{III},\ A^{IV},\ A^{V},\ \&c.$$

huius ſeries primarum differentiarum erit ipſa ſeries propoſita:

$$a,\ a^{I},\ a^{II},\ a^{III},\ a^{IV},\ a^{V},\ \&c.$$

55. Hanc ob rem ſeriei propoſitae differentiae primae, erunt differentiae ſecundae ſummatricis, atque differentiae ſecundae illius erunt differentiae tertiae huius, tertiae autem illius quartae huius, atque ita porro. Quare, ſi ſeries propoſita tandem habeat differentias conſtantes, tunc etiam eius ſummatrix ad differentias conſtantes deducetur, eritque igitur ſeries eiuſdem naturae, at uno ordine ſuperior. Huiuſmodi ergo ſerierum perpetuo terminus ſummatorius exhiberi poterit per expreſſionem finitam. Namque terminus generalis ſeriei:

$$0,\ A,\ A^{I},\ A^{II},\ A^{III},\ A^{IV},\ \&c.$$

ſeu is, qui indici x convenit exhibebit ſummam $x - 1$ terminorum ſeriei huius a, a^{I}, a^{II}, a^{III}, a^{IV}, &c. atque ſi tum loco x ſcribatur $x + 1$, orietur ſumma x terminorum, ipſeque terminus ſummatorius.

56. Sit igitur ſeriei propoſitae

a, a^{I}, a^{II}, a^{III}, a^{IV}, a^{V}, a^{VI}, &c.

Series differentiarum primarum

b, b^{I}, b^{II}, b^{III}, b^{IV}, b^{V}, b^{VI}, &c.

Series differentiarum ſecundarum

c, c^{I}, c^{II}, c^{III}, c^{IV}, c^{V}, c^{VI}, &c.

Series differentiarum tertiarum

d, d^{I}, d^{II}, d^{III}, d^{IV}, d^{V}, d^{VI}, &c.

ſicque porro donec ad differentias conſtantes perveniatur. Deinde formetur ſeries ſummatrix, quae cum praefixa o in locum termini primi, cum ſuis differentiis continuis ſe habebit ſequenti modo:

INDICES.

1, 2, 3, 4, 5, 6, 7, &c.

SUMMATRIX.

0, A, A^{I}, A^{II}, A^{III}, A^{IV}, A^{V}, &c.

SERIES PROPOSITA.

	a, a^{I}, a^{II}, a^{III}, a^{IV}, a^{V}, a^{VI}, &c.
Differ. I.	b, b^{I}, b^{II}, b^{III}, b^{IV}, b^{V}, b^{VI}, &c.
Differ. II.	c, c^{I}, c^{II}, c^{III}, c^{IV}, c^{V}, c^{VI}, &c.
Differ. III.	d, d^{I}, d^{II}, d^{III}, d^{IV}, d^{V}, d^{VI}, &c.

erit ſeriei ſummatricis terminus generalis, ſeu qui indici x reſpondet

$$0+(x-1)a+\frac{(x-1)(x-2)}{1\,.\,2}b+\frac{(x-1)(x-2)(x-3)}{1\,.\,2\,.\,3}c+\&c.$$

qui ſimul exhibet ſummam $x-1$ terminorum ſeriei propoſitae, a, a^{I}, a^{II}, a^{III}, a^{IV}, &c.

57. Quod ſi ergo in hac ſumma loco $x-1$ ſcribatur x, prodibit ſeriei propoſitae terminus ſummatorius ſummam x terminorum complectens

$$= xa + \frac{x(x-1)}{1.\ 2} b + \frac{x(x-1)(x-2)}{1.\ 2.\ 3} c + \frac{x(x-1)(x-2)(x-3)}{1.\ 2.\ 3.\ 4} d + \&c.$$

Hinc, ſi litterae b, c, d, e, valores ipſis aſſignatos retineant, erit

SERIEI.

$$a,\ a^{I},\ a^{II},\ a^{III},\ a^{IV},\ a^{V},\ \&c.$$

TERMINUS GENERALIS.

$$a + (x-1)b + \frac{(x-1)(x-2)}{1.\ 2} c + \frac{(x-1)(x-2)(x-3)}{1.\ 2.\ 3} d + \frac{(x-1)(x-2)(x-3)(x-4)}{1.\ 2.\ 3.\ 4} e + \&c.$$

ET TERMINUS SUMMATORIUS.

$$xa + \frac{x(x-1)}{1.\ 2} b + \frac{x(x-1)(x-2)}{1.\ 2.\ 3} c + \frac{x(x-1)(x-2)(x-3)}{1.\ 2.\ 3.\ 4} d + \&c.$$

Invento ergo ſeriei cuiuſvis ordinis hoc, quem oſtendimus, modo termino generali, non difficulter ex eo terminus ſummatorius reperietur, quippe qui ex iiſdem differentiis conflatur.

58. Hic modus terminum ſummatorium per differentias ſeriei inveniendi imprimis ad eiuſmodi ſeries, quae tandem ad differentias conſtantes deducunt, eſt accomodatus; in aliis enim caſibus expreſſio finita non reperitur. Quodſi autem ea, quae ante de indole termini ſummatorii ſunt expoſita, attentius perpendamus, alius modus ſe offert terminum ſummatorium immediate ex termino generali inveniendi, qui multo latius patet, atque in infinitis caſibus ad expreſſiones finitas deducit, quibus prior modus infinitas exhibet. Sit enim propoſita ſeries quaecunque.

$$a,\ b,\ c,\ d,\ e,\ f,\ \&c.$$

cuius terminus generalis, ſeu indici x reſpondens ſit $= X$; terminus autem ſummatorius ſit $= S$, qui cum ſummam tot terminorum ab initio exhibeat, quot numerus x continet unitates, erit ſumma $x - 1$ terminorum $= S - X$; eritque adeo

adeo X differentia expreſſionis $S - X$, cum relinquatur, ſi haec a ſequente S ſubtrahatur.

59. Cum igitur ſit $X = \Delta(S - X)$ differentia eo modo ſumta, quem capite praecedente docuimus, hoc tantum diſcrimine, ut quantitas illa conſtans ω hic nobis ſit $= 1$. Quare, ſi ad ſummas regrediamur, erit $\Sigma X = S - X$, ideoque terminus ſummatorius quaeſitus

$$S = \Sigma X + X + C.$$

Quaeri ergo debet ſumma functionis X methodo ante tradita, ad eamque addi ipſe terminus generalis X, eritque aggregatum terminus ſummatorius. Quoniam autem in ſummis ſumendis involvitur quantitas conſtans, ſive addenda ſive ſubtrahenda; ea ad praeſentem caſum accomodari debebit. Manifeſtum autem eſt, ſi ponatur $x = 0$, quo caſu numerus terminorum ſummandorum eſt nullus, ſummam quoque fore nullam; ex quo quantitas illa conſtans C ita determinari debebit, ut poſitò $x = 0$, fiat quoque $S = 0$. Poſitis ergo in illa aequatione $S = \Sigma X + X + C$ tam $S = 0$, quam $x = 0$, valor ipſius C invenietur.

60. Quoniam ergo hic totum negotium ad ſummationem functionum ſupra monſtratam reducitur, ponendo $\omega = 1$, exinde depromamus ſummationes traditas; ac primo quidem pro poteſtatibus ipſius x erit

$$\Sigma x^0 = \Sigma 1 = x$$

$$\Sigma x = \tfrac{1}{2}x^2 - \tfrac{1}{2}x$$

$$\Sigma x^2 = \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{6}x$$

$$\Sigma x^3 = \tfrac{1}{4}x^4 - \tfrac{1}{2}x^3 + \tfrac{1}{4}x^2$$

$$\Sigma x^4 = \tfrac{1}{5}x^5 - \tfrac{1}{2}x^4 + \tfrac{1}{3}x^3 - \tfrac{1}{30}x$$

$$\Sigma x^5 = \tfrac{1}{6}x^6 - \tfrac{1}{2}x^5 + \tfrac{5}{12}x^4 - \tfrac{1}{12}x^2$$

$$\Sigma x^6 = \tfrac{1}{7}x^7 - \tfrac{1}{2}x^5 + \tfrac{1}{2}x^5 - \tfrac{1}{6}x^3 + \tfrac{1}{42}x$$

quibus accenſeatur ſummatio generalis poteſtatis x^n §. 29. tradita, dummodo ibi ubique loco ω unitas ſcribatur. Harum ergo formularum ope omnium ſerierum, quarum termi-

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ni generales ſunt functiones rationales integrae ipſius x, termini ſummatorii expedite inveniri poterunt.

61. Denotet S. X terminum ſummatorium ſeriei, cuius terminus generalis eſt $= X$; eritque, ut vidimus,

$$S.X = \Sigma X + X + C$$

dummodo conſtans C ita aſſumatur, ut terminus ſummatorius S. X evaneſcat poſito $x = 0$. Hinc igitur terminos ſummatorios ſerierum poteſtatum, ſeu quarum termini generales comprehenduntur in hac forma x^n exprimamus. Poſito itaque

$$S.x^n = 1 + 2^n + 3^n + 4^n + \ldots + x^n$$

erit

$$S.x^n = \frac{1}{n+1}x^{n+1} + \tfrac{1}{2}x^n + \tfrac{1}{2}.\frac{n}{2.3}x^{n-1} - \tfrac{1}{6}.\frac{n(n-1)(n-2)}{2.\,3.\,4.\,5}x^{n-3}$$

$$+ \tfrac{1}{6}.\frac{n(n-1)(n-2)(n-3)(n-4)}{2.\,3.\,4.\,5.\,6.\,7.}x^{n-5} - \tfrac{3}{10}.\frac{n(n-1)\;.\;.\;(n-6)}{2.\,3\;.\;.\;.\;8.\,9}x^{n-7}$$

$$+ \tfrac{5}{6}.\frac{n(n-1)\;.\;.\;(n-8)}{2.\,3.\,4\;.\;.\;10.11}x^{n-9} - \tfrac{691}{210}.\frac{n(n-1)\;.\;.\;(n-10)}{2.\,3\;.\;.\;12.\,13}x^{n-11}$$

$$+ \tfrac{35}{2}.\frac{n(n-1)\;.\;.\;.\;(n-12)}{2.\,3\;.\;.\;.\;14.15}x^{n-13} - \tfrac{3617}{30}.\frac{n(n-1)\;.\;.\;(n-14)}{2.3\;.\;.\;.\;16.17}x^{n-15}$$

$$+ \frac{43867}{42}.\frac{n(n-1)\;.\;.\;.\;.\;.\;.\;.\;(n-16)}{2.\;3\;.\;.\;.\;.\;.\;.\;.\;18.\;19}x^{n-17}$$

$$- \frac{1222277}{110}.\frac{n(n-1)\;.\;.\;.\;.\;.\;.\;(n-18)}{2.\;3\;.\;.\;.\;.\;.\;.\;20.\;21}x^{n-19}$$

$$+ \frac{854513}{6}.\frac{n(n-1)\;.\;.\;.\;.\;.\;.\;.\;(n-20)}{2.\;3\;.\;.\;.\;.\;.\;.\;22.\;23}x^{n-21}$$

$$- \frac{1181820455}{546}.\frac{n(n-1)\;.\;.\;.\;.\;.\;(n-22)}{2.\;3\;.\;.\;.\;.\;.\;24.\;25}x^{n-23}$$

$$+ \frac{76977927}{2}.\frac{n(n-1)\;.\;.\;.\;.\;.\;.\;(n-24)}{2.\;3\;.\;.\;.\;.\;.\;.\;26.\;27}x^{n-25}$$

&c.

62.

62. Hinc ergo ſummae pro variis ipſius n valoribus ita ſe habebunt.

$$S.x^0 = x$$
$$S.x^1 = \tfrac{1}{2}x^2 + \tfrac{1}{2}x$$
$$S.x^2 = \tfrac{1}{3}x^3 + \tfrac{1}{2}xx + \tfrac{1}{6}x$$
$$S.x^3 = \tfrac{1}{4}x^4 + \tfrac{1}{2}x^3 + \tfrac{1}{4}x^2$$
$$S.x^4 = \tfrac{1}{5}x^5 + \tfrac{1}{2}x^4 + \tfrac{1}{3}x^3 - \tfrac{1}{30}x$$
$$S.x^5 = \tfrac{1}{6}x^6 + \tfrac{1}{2}x^5 + \tfrac{5}{12}x^4 - \tfrac{1}{12}x^2$$
$$S.x^6 = \tfrac{1}{7}x^7 + \tfrac{1}{2}x^6 + \tfrac{1}{2}x^5 - \tfrac{1}{6}x^3 + \tfrac{1}{42}x$$
$$S.x^7 = \tfrac{1}{8}x^8 + \tfrac{1}{2}x^7 + \tfrac{7}{12}x^6 - \tfrac{7}{24}x^4 + \tfrac{1}{12}x^2$$
$$S.x^8 = \tfrac{1}{9}x^9 + \tfrac{1}{2}x^8 + \tfrac{2}{3}x^7 - \tfrac{7}{15}x^5 + \tfrac{2}{9}x^3 - \tfrac{1}{30}x$$
$$S.x^9 = \tfrac{1}{10}x^{10} + \tfrac{1}{2}x^9 + \tfrac{3}{4}x^8 - \tfrac{7}{10}x^6 + \tfrac{1}{2}x^4 - \tfrac{3}{20}x^2$$
$$S.x^{10} = \tfrac{1}{11}x^{11} + \tfrac{1}{2}x^{10} + \tfrac{5}{6}x^9 - x^7 + x^5 - \tfrac{1}{2}x^3 + \tfrac{5}{66}x$$
$$S.x^{11} = \tfrac{1}{12}x^{12} + \tfrac{1}{2}x^{11} + \tfrac{11}{12}x^{10} - \tfrac{11}{8}x^8 + \tfrac{11}{6}x^6 - \tfrac{11}{8}x^4 + \tfrac{5}{12}x^2$$
$$S.x^{12} = \tfrac{1}{13}x^{13} + \tfrac{1}{2}x^{12} + x^{11} - \tfrac{11}{6}x^9 + \tfrac{22}{7}x^7 - \tfrac{33}{10}x^5 + \tfrac{5}{3}x^3 - \tfrac{691}{2730}x$$
$$S.x^{13} = \tfrac{1}{14}x^{14} + \tfrac{1}{2}x^{13} + \tfrac{13}{12}x^{12} - \tfrac{143}{60}x^{10} + \tfrac{143}{28}x^8 - \tfrac{143}{20}x^6 + \tfrac{65}{12}x^4 - \tfrac{691}{420}x^2$$
$$S.x^{14} = \tfrac{1}{15}x^{15} + \tfrac{1}{2}x^{14} + \tfrac{7}{6}x^{13} - \tfrac{91}{30}x^{11} + \tfrac{143}{18}x^9 - \tfrac{143}{10}x^7 + \tfrac{91}{6}x^5 - \tfrac{691}{90}x^3 + \tfrac{7}{6}x$$
$$S.x^{15} = \tfrac{1}{16}x^{16} + \tfrac{1}{2}x^{15} + \tfrac{5}{4}x^{14} - \tfrac{91}{24}x^{12} + \tfrac{143}{12}x^{10} - \tfrac{429}{16}x^8 + \tfrac{455}{12}x^6 - \tfrac{691}{24}x^4 + \tfrac{35}{4}x^2$$
$$S.x^{16} = \tfrac{1}{17}x^{17} + \tfrac{1}{2}x^{16} + \tfrac{4}{3}x^{15} - \tfrac{14}{3}x^{13} + \tfrac{52}{3}x^{11} - \tfrac{143}{3}x^9 + \tfrac{260}{3}x^7 - \tfrac{1382}{15}x^5 + \tfrac{140}{3}x^3 - \tfrac{3617}{510}x$$

&c.

quae ſummae ex forma generali uſque ad poteſtatem vigeſi-

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mam nonam continuari poſſunt. Atque adhuc ulterius progredi liceret, ſi coefficientes illi numerici ulterius eſſent eruti.

63. Ceterum, in his formulis lex quaedam obſervatur, cuius ope quaelibet ex praecedente facile inveniri poteſt, excepto tantum termino ultimo, ſi in eo poteſtas ipſius x prima contineatur: tum enim in ſumma ſequente unus terminus inſuper accedit. Hoc autem omiſſo, ſi fuerit

$$S.x^n = \alpha x^{n+1} + \beta x^n + \gamma x^{n-1} - \delta x^{n-3} + \varepsilon x^{n-5} - \zeta x^{n-7} + \eta x^{n-9} - \&c.$$

erit ſequens ſumma:

$$S.x^{n+1} = \frac{n+1}{n+2}\alpha x^{n+2} + \frac{n+1}{n+1}\beta x^{n+1} + \frac{n+1}{n}\gamma x^n - \frac{n+1}{n-2}\delta x^{n-2} + \frac{n+1}{n-4}\varepsilon x^{n-4} - \frac{n+1}{n-6}\zeta x^{n-6} + \frac{n+1}{n-8}\eta x^{n-8} - \&c.$$

unde ſi n fuerit numerus par, ſequens ſumma vera prodit: at ſi n fuerit numerus impar, tum in ſequente ſumma praeterea deſiderabitur terminus ultimus, cuius forma erit $\pm \phi x$. Interim tamen hic ſine aliis ſubſidiis ita inveniri poterit. Cum enim ſi ponatur $x = 1$, ſumma unici tantum termini, (hoc eſt terminus primus, qui erit $= 1$) oriri debeat: ponatur in omnibus terminis iam inventis $x = 1$, ipſaque ſumma ſtatuatur $= 1$, quo facto valor ipſius ϕ elicietur, eoque invento ulterius progredi licebit. Atque hoc pacto omnes iſtae ſummae inveniri potuiſſent. Sic, cum ſit

$$S.x^5 = \tfrac{1}{6}x^6 + \tfrac{1}{2}x^5 + \tfrac{5}{12}x^4 - \tfrac{1}{12}x^2$$

erit $S.x^6 = \frac{6}{7}\cdot\frac{1}{6}x^7 + \frac{6}{6}\cdot\frac{1}{2}x^6 + \frac{6}{5}\cdot\frac{5}{12}x^5 - \frac{6}{3}\cdot\frac{1}{12}x^3 + \phi x$

ſeu $S.x^6 = \frac{1}{7}x^7 + \frac{1}{2}x^6 + \frac{1}{2}x^5 - \frac{1}{6}x^3 + \phi x$.

Ponatur nunc $x = 1$, fiet $1 = \frac{1}{7} + \frac{1}{2} + \frac{1}{2} - \frac{1}{6} + \phi$ ideoque $\phi = \frac{1}{6} - \frac{1}{7} = \frac{1}{42}$, uti ex forma generali invenimus.

64. Ope harum formularum ſummatoriarum nunc facile omnium ſerierum, quarum termini generales ſunt functiones ipſius x rationales integrae, termini ſummatorii inveniri po-

poterunt, hocque multo expeditius, quam praecedente methodo per differentias.

EXEMPLUM I.

Invenire terminum ſummatorium huius ſeriei

2, 7, 15, 26, 40, 57, 77, 100, 126, &c.

cuius terminus generalis eſt $\frac{3xx+x}{2}$.

Cum terminus generalis conſtet duobus membris, quaeratur pro utroque terminus ſummatorius ex formulis ſuperioribus

$$S.\tfrac{1}{2}xx = \tfrac{1}{6}x^3 + \tfrac{1}{4}xx + \tfrac{1}{12}x$$

& $$S.\tfrac{1}{2}x = \ldots \tfrac{1}{4}xx + \tfrac{1}{4}x$$

eritque $S.\frac{3xx+x}{2} = \frac{1}{2}x^3 + xx + \frac{1}{2}x = \frac{1}{2}x(x+1)^2$

qui eſt terminus ſummatorius quaeſitus. Sic, ſi ponatur $x = 5$, erit $\frac{5}{2}.6^2 = 90$, ſumma quinque terminorum

$$2+7+15+26+40 = 90.$$

EXEMPLUM II.

Invenire terminum ſummatorium ſeriei

1, 27, 125, 343, 729, 1331, &c.

quae continet cubos numerorum imparium.

Terminus generalis huius ſeriei eſt

$$= (2x-1)^3 = 8x^3 - 12xx + 6x - 1,$$

unde terminus ſummatorius ſequenti modo colligetur.

$$+ 8.\ S.x^3 = 2x^4 + 4x^3 + 2x^2$$

& $$- 12.\ S.x^2 = .\ - 4x^3 - 6x^2 - 2x$$

atque $$+ 6.\ S.x = \ldots + 3x^2 + 3x$$

denique $$- 1.\ S.x^0 = \ldots\ldots - x.$$

Erit ſcilicet ſumma quaeſita $= 2x^4 - x^2 = xx(2xx-1)$.
Uti, ſi ponatur $x=6$ erit $36.71 = 2556$ ſumma ſex terminorum ſeriei propoſitae $= 1+27+125+343+729+1331 = 2556$.

65. Quod ſi terminus generalis fuerit productum ex

factoribus ſimplicibus, tum terminus ſummatorius facilius reperietur per ea, quae ſupra §. 32. & ſequentibus ſunt tradita. Cum enim, poſito $\omega = 1$, ſit

$$\Sigma\,(x+n) = \tfrac{1}{2}(x+n-1)(x+n)$$

& $$\Sigma\,(x+n)(x+n+1) = \tfrac{1}{3}(x+n-1)(x+n)(x+n+1)$$

& $$\Sigma\,(x+n)(x+n+1)(x+n+2) = \tfrac{1}{4}(x+n-1)(x+n)(x+n+1)(x+n+2)$$

&c.

ſi ad has ſummas ipſos terminos generales addamus, ſimulque conſtantem adiiciamus, quae poſito $x = 0$, reddat terminum ſummatorium evaneſcentem, ſequentes obtinebimus terminos ſummatorios

$$S.(x+n) = \tfrac{1}{2}(x+n)(x+n+1) - \tfrac{1}{2}n(n+1)$$

& $$S.(x+n)(x+n+1) = \tfrac{1}{3}(x+n)(x+n+1)(x+n+2) - \tfrac{1}{3}n(n+1)(n+2)$$

& $$S.(x+n)(x+n+1)(x+n+2) = \tfrac{1}{4}(x+n)(x+n+1)(x+n+2)(x+n+3) - \tfrac{1}{4}n(n+1)(n+2)(n+3)$$

ſicque porro.

Si ergo fuerit vel $n = 0$ vel $n = -1$, quantitas conſtans in his ſummis evaneſcit.

66. Seriei ergo 1, 2, 3, 4, 5, &c. cuius terminus generalis eſt $= x$, terminus ſummatorius erit $= \tfrac{1}{2}x(x+1)$ ſerieſque ſummatrix haec: 1, 3, 6, 10, 15, &c. cuius porro terminus ſummatorius erit $= \frac{x(x+1)(x+2)}{1 \cdot 2 \cdot 3}$, & ſeries ſummatrix haec: 1, 4, 10, 20, 35, &c. Haec vero denuo terminum ſummatorium habebit $= \frac{x(x+1)(x+2)(x+3)}{1 \cdot 2 \cdot 3 \cdot 4}$, qui erit terminus generalis ſeriei 1, 5, 15, 35, 70, &c. huiuſque terminus ſummatorius erit $= \frac{x(x+1)(x+2)(x+3)(x+4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$. Hae autem ſeries prae reliquis probe ſunt notandae, quoniam earum ubique ampliſſimus eſt uſus. Ex his enim deſumuntur coefficientes binomii ad dignitates elevati, qui quam late pateant, cuique in his rebus parum verſato abunde conſtat.

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67. Ex his etiam illi termini ſummatorii, quos ante ex differentiis elicuimus, facile inveniuntur. Cum enim ibi terminum generalem ſequenti forma invenerimus expreſſum

$$a+\frac{(x-1)}{1}b+\frac{(x-1)(x-2)}{1\,.\,2}c+\frac{(x-1)(x-2)(x-3)}{1\,.\,2\,.\,3}d+\&c.$$

ſi cuiuſque membri terminum ſummatorium quaeramus eoſque omnes addamus, habebimus terminum ſummatorium huic termino generali convenientem. Sic cum ſit

$$S\;1 = x$$

$$\&\quad S\,(x-1) = \tfrac{1}{2}\,x\,(x-1)$$

$$\text{atque}\quad S\,(x-1)(x-2) = \tfrac{1}{3}\,x\,(x-1)(x-2)$$

$$\&\quad S\,(x-1)(x-2)(x-3) = \tfrac{1}{4}\,x\,(x-1)(x-2)(x-3)$$

&c.

erit terminus ſummatorius quaeſitus:

$$xa+\frac{x(x-1)}{1\,.\,2}b+\frac{x(x-1)(x-2)}{1\,.\,2\,.\,3}c+\frac{x(x-1)(x-2)(x-3)}{1\,.\,2\,.\,3\,.\,4}d+\&c$$

quae forma non diſcrepat ab ea, quam ante ex differentiis obtinuimus.

68. Deinde etiam haec terminorum ſummatoriorum inventio ad fractiones accomodari poteſt: quia enim ſupra §. 34. invenimus eſſe, ponendo $\omega = 1$

$$\Sigma\frac{1}{(x+n)(x+n+1)} = -1\,.\,\frac{1}{x+n}$$

erit

$$S.\frac{1}{(x+n)(x+n+1)} = -1\,.\,\frac{1}{x+n+1}+\frac{1}{n+1}$$

Simili modo, ſi ad ſummas ſupra inventas ipſos terminos generales addamus, ſeu quod idem eſt, ſi in illis expreſſionibus loco x ponamus $x+1$ habebimus

$$S.\frac{1}{(x+n)(x+n+1)(x+n+2)} = -\frac{1}{2}\,.\,\frac{1}{(x+n+1)(x+n+2)} + \frac{1}{2}\,.\,\frac{1}{(n+1)(n+2)}$$

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$$\& \quad S.\frac{1}{(x+n)(x+n+1)(x+n+2)(x+n+3)} =$$

$$-\frac{1}{3}.\frac{1}{(x+n+1)(x+n+2)(x+n+3)} + \frac{1}{3}.\frac{1}{(n+1)(n+2)(n+3)}$$

quae formae facile pro lubitu ulterius continuantur.

69. Quia erit $S.\frac{1}{(x+n)(x+n+1)} = \frac{1}{n+1} - \frac{1}{x+n+1}$

erit quoque $S.\frac{1}{x+n} - S.\frac{1}{x+n+1} = \frac{1}{n+1} - \frac{1}{x+n+1}$; etſi ergo neuter horum duorum terminorum ſummatoriorum ſeorſim exhiberi poteſt, tamen eorum differentia cognoſcitur; hincque in pluribus caſibus ſummae ſerierum ſatis expedite aſſignantur: id quod uſu venit, ſi terminus generalis fuerit fractio, cuius denominator in factores ſimplices reſolvi poteſt. Tum enim tota fractio in fractiones partiales reſolvatur; quo facto, ope huius lemmatis mox patebit, utrum terminus ſummatorius exhiberi queat nec ne.

EXEMPLUM I.

Invenire terminum ſummatorium ſeriei huius:

$$1 + \tfrac{1}{3} + \tfrac{1}{6} + \tfrac{1}{10} + \tfrac{1}{15} + \tfrac{1}{21} + \&c.$$

cuius terminus generalis eſt $= \frac{2}{xx+x}$.

Terminus iſte generalis per reſolutionem reducitur ad hanc formam $\frac{2}{x} - \frac{2}{x+1}$. Hinc terminus ſummatorius erit $= 2\,S.\frac{1}{x} - 2\,S.\frac{1}{x+1}$, qui ergo per praecedens lemma erit $= 2 - \frac{2}{x+1} = \frac{2x}{x+1}$. Sic, ſi ſit $x = 4$, erit $\tfrac{8}{5} = 1 + \tfrac{1}{3} + \tfrac{1}{6} + \tfrac{1}{10}$.

EXEMPLUM II.

Quaeratur terminus ſummatorius ſeriei huius: $\tfrac{1}{5}, \tfrac{1}{21}, \tfrac{1}{45}, \tfrac{1}{77}, \tfrac{1}{117}$, &c.

cuius terminus generalis eſt $= \frac{1}{4xx+4x-3}$.

Quia termini generalis denominator habet factores $2x-1$ & $2x+3$, is reſolvetur in has partes:

$$\frac{1}{4}\cdot\frac{1}{2x-1}-\frac{1}{4}\cdot\frac{1}{2x+3}=\frac{1}{8}\cdot\frac{1}{x-\frac{1}{2}}-\frac{1}{8}\cdot\frac{1}{x+\frac{3}{2}}.$$

At eſt $S.\frac{1}{x-\frac{1}{2}}=S.\frac{1}{x+\frac{1}{2}}+2-\frac{1}{x+\frac{1}{2}}$

& $S.\frac{1}{x+\frac{1}{2}}=S.\frac{1}{x+\frac{3}{2}}+\frac{2}{3}-\frac{1}{x+\frac{3}{2}}$

ergo $S.\frac{1}{x-\frac{1}{2}}-S.\frac{1}{x+\frac{3}{2}}=2+\frac{2}{3}-\frac{1}{x+\frac{1}{2}}-\frac{1}{x+\frac{3}{2}}$

cuius pars octava dabit terminum ſummatorium quaeſitum nempe

$$\frac{1}{4}+\frac{1}{12}-\frac{1}{8x+4}-\frac{1}{8x+12}=\frac{x}{4x+2}+\frac{x}{3(4x+6)}=\frac{x(4x+5)}{3(2x+1)(2x+3)}.$$

70. Quoniam numeri figurati, quos coefficientes binomii ad dignitates evecti praebent, prae ceteris notari merentur, ſummas ſerierum exhibeamus, quarum numeratores ſint $=1$, denominatores vero numeri figurati; id quod ex §. 68. facile fiet. Seriei ergo cuius

Terminus generalis eſt	Terminus ſummatorius erit
$\frac{1\cdot 2}{x(x+1)}$	$\frac{2}{1}-\frac{2}{x+1}$
$\frac{1\cdot 2\cdot 3}{x(x+1)(x+2)}$	$\frac{3}{2}-\frac{1\cdot 3}{(x+1)(x+2)}$
$\frac{1\cdot 2\cdot 3\cdot 4}{x(x+1)(x+2)(x+3)}$	$\frac{4}{3}-\frac{1\cdot 2\cdot 4}{(x+1)(x+2)(x+3)}$
$\frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{x(x+1)(x+2)(x+3)(x+4)}$	$\frac{5}{4}-\frac{1\cdot 2\cdot 3\cdot 5}{(x+1)(x+2)(x+3)(x+4)}$
&c.	&c.

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unde lex, qua iſtae expreſſiones progrediuntur, ſponte apparet. Neque vero hinc terminus ſummatorius, qui conveniat termino generali $\frac{1}{x}$, colligi poteſt, quippe qui per formulam definitam exprimi nequit.

71. Quoniam terminus ſummatorius praebet ſummam tot terminorum, quot unitates continentur in indice x; manifeſtum eſt harum ſerierum in infinitum continuatarum ſummas obtineri, ſi ponatur index x infinitus: quo caſu expreſſionum modo inventarum termini poſteriores, ob denominatores in infinitum abeuntes, evaneſcent.
Hinc iſtae ſeries infinitae finitas habebunt ſummas, quae erunt

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \&c. = \frac{2}{1}$$
$$1 + \frac{1}{4} + \frac{1}{10} + \frac{1}{20} + \frac{1}{35} + \&c. = \frac{3}{2}$$
$$1 + \frac{1}{5} + \frac{1}{15} + \frac{1}{35} + \frac{1}{70} + \&c. = \frac{4}{3}$$
$$1 + \frac{1}{6} + \frac{1}{21} + \frac{1}{56} + \frac{1}{126} + \&c. = \frac{5}{4}$$
$$1 + \frac{1}{7} + \frac{1}{28} + \frac{1}{84} + \frac{1}{210} + \&c. = \frac{6}{5}$$
&c.

Omnium ergo ſerierum, quarum termini ſummatorii habentur, in infinitum continuatarum ſummae exhiberi poterunt poſito $x = \infty$, dummodo hoc caſu ſummae fiant finitae: quod quidem evenit, ſi in termino ſummatorio x tot habeat dimenſiones in denominatore, quot habet in numeratore.

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CAPUT III.
DE INFINITIS ATQUE INFINITE PARVIS.

72.

Cum omnis Quantitas, quantumvis ſit magna, ulterius augeri poſſit, neque quicquam obſtet, quominus ad datam quantitatem quamcumque alia quantitas eiusdem generis addi queat; omnis quoque quantitas ſine fine augeri poterit: neque enim unquam tam magna fiet, ut ipſi nihil amplius adiici poſſit. Nulla igitur datur quantitas tam magna, qua maior concipi nequeat: hincque extra dubium erit poſitum, *omnem quantitatem in infinitum augeri poſſe*. Qui enim hoc negaverit, is affirmare cogitur, dari limitem, quem quantitas, cum attigerit, ſuperare nequeat, atque ideo ſtatuere debebit quantitatem, cui nihil amplius adiici poſſit; quod cum ſit abſurdum atque quantitatis notioni adverſetur, neceſſario concedendum eſt, omnem quantitatem ſine fine continuo magis, hoc eſt, in infinitum augeri poſſe.

73. In ſingulis quantitatum ſpeciebus hoc etiam clarius perſpicietur. Sic, nemo facile reperietur, qui ſtatuerit ſeriem numerorum naturalium 1, 2, 3, 4, 5, 6, &c. ita uſquam eſſe determinatam, ut ulterius continuari non poſſit. Nullus enim datur numerus, ad quem non inſuper unitas addi, ſicque numerus ſequens maior exhiberi queat; hinc ſeries numerorum naturalium ſine fine progreditur, neque unquam pervenitur ad numerum maximum, quo maior prorſus non detur. Simili modo linea recta nunquam eouſque produci poteſt, ut inſuper ulterius prolongari non poſſit. Quibus evincitur, tam numeros in infinitum augeri, quam lineas in infinitum produci poſſe. Quae cum ſint ſpecies quantitatum,

 ſimul

ſimul intelligitur, omni quantitate, quantumvis ſit magna, adhuc dari maiorem, hacque denuo maiorem, ſicque augendo continuo ulterius ſine fine, hoc eſt in infinitum, procedi poſſe.

74. Quanquam autem haec ſunt adeo perſpicua, ut qui ea negare vellet, ſibi ipſe contradicere deberet; tamen iſta infiniti doctrina a pluribus, qui eam explicare ſunt conati, tantopere eſt offuſcata, tantiſque difficultatibus atque etiam contradictionibus obvoluta, ut, qua ſe extricarent, nulla via pateret. Ex eo, quod quantitas in infinitum augeri poſſit, quidam concluſerunt, dari revera quantitatem infinitam, eamque ita deſcripſerunt, ut nullum amplius augmentum ſuſcipere poſſit. Hoc autem ipſo ideam quantitatis evertunt, dum eiuſmodi quantitatem ſtatuunt, quae ulterius augeri nequeat. Praeterea vero ſecum ipſi infinitum admittentes pugnant; dum enim incrementi, quo quantitas ſit capax, finem faciunt, ſimul negant quantitatem ſine fine augeri poſſe, negant ergo quoque quantitatem in infinitum augeri poſſe, quoniam utraque locutio congruit: ſicque, dum quantitatem infinitam ſtatuunt, eam ſimul tollunt. Si enim quantitas ſine fine, hoc eſt in infinitum, augeri nequeat, certe nulla quantitas infinita exiſtere poterit.

75. Hinc igitur ex eo ipſo, quod omnis quantitas in infinitum augeri poſſit, ſequi videtur nullam dari quantitatem infinitam. Quantitas enim continuis incrementis aucta, infinita non evadet, niſi iam ſine fine increverit: quod autem ſine fine fieri debet, id non tanquam iam factum concipi poteſt. Interim tamen non ſolum huiuſmodi quantitatem, ad quam incrementis ſine fine congeſtis pervenitur, certo charactere indicare, ſicque debito modo in calculum inducere licet, uti mox fuſius oſtendemus; ſed etiam in mundo eiuſmodi caſus exiſtere, vel ſaltem concipi poſſunt, quibus numerus infinitus actu exiſtere videatur. Sic ſi materia in infinitum ſit diviſibilis, uti plures Philoſophi ſtatuerunt, numerus

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partium, quibus datum quodque materiae fruſtum conſtat, revera erit infinitus; ſi enim ſtatueretur finitus, materia certe non in infinitum foret diviſibilis. Simili modo ſi univerſus mundus eſſet infinitus, uti pluribus placuit, numerus corporum mundum componentium finitus certe eſſe non poſſet, foretque ideo quoque infinitus.

76. Haec etiamſi inter ſe pugnare videantur, tamen ſi attentius perpendantur, a cunctis incommodis liberari poterunt. Qui enim ſtatuit materiam in infinitum eſſe diviſibilem; is negat in diviſione materiae continua unquam ad partes tam parvas perveniri, quae ulterius dividi nequeant: nullas ergo materia habebit partes, ulterius individuas; cum ſingulae particulae, ad quas per continuam diviſionem iam ſit perventum, ulterius ſe ſubdividi patiantur. Qui igitur dicit hoc caſu numerum partium fore infinitum, is partes ultimas, quae ulterius ſint individuae, intelligit; ad quas cum nunquam perveniatur, & quae propterea nullae ſunt, is has ipſas partes, quae nullae ſunt, numerare conatur. Si enim materia ſine fine continuo ulterius ſubdividi poteſt, partibus individuis ſeu ſimplicibus prorſus caret: neque adeo quicquam ſupereſt, quod numerari queat. Hanc ob rem qui materiam in infinitum diviſibilem ſtatuit, is ſimul negat, materiam ex partibus ſimplicibus eſſe compoſitam.

77. Quod ſi autem, dum de partibus alicuius corporis ſeu materiae loquimur, non ultimas ſeu ſimplices, quippe quae nullae ſunt, intelligamus, ſed eas, quas diviſio revera produxit; tum admiſſa hac hypotheſi de diviſibilitate materiae in infinitum, unumquodque vel minimum materiae fruſtum non ſolum in plurimas partes diſſecari, ſed etiam nullus numerus tam magnus aſſignari poterit, quo non maior partium ex illo fruſto ſectarum numerus exhiberi queat. Numerus ergo partium non quidem ultimarum, ſed quae ipſae adhuc ſint ulterius diviſibiles, quae unumquodque corpus componunt, omni numero aſſignabili erit maior. Simili modo,

do, ſi univerſus mundus ſit infinitus, numerus corporum mundum conſtituentium pariter omni aſſignabili erit maior; qui cum finitus eſſe nequeat, ſequitur numerum infinitum & numerum omni aſſignabili maiorem eſſe nomina ſynonyma.

78. Qui ergo hoc modo diviſibilitatem materiae in infinitum intuetur, nullis incommodis, quae vulgo huic opinioni imputantur, ſe implicat, nihilque affirmare cogitur, quod ſanae rationi adverſetur. Qui autem contra materiam in infinitum diviſibilem eſſe negant, ii in maximas difficultates prolabuntur, ex quibus ſe nullo prorſus modo extrahere poſſunt. Statuere enim coguntur unumquodque corpus nonniſi in certum partium numerum diſſecari poſſe, ad quas ſi fuerit perventum, nulla diviſio ulterior locum inveniat; quas ultimas particulas alii *atomos*, alii *monades* atque *entia ſimplicia* vocant. Cur autem iſtae ultimae particulae nullam amplius diviſionem admittant, duplex eſſe poteſt cauſa: altera, quod omni extenſione careant; altera quod quidem ſint extenſae, ſed tamen tam durae atque ita comparatae, ut nulla vis ad eas diſſecandas ſufficiat. Utrumvis patroni huius opinionis dicant, ſeſe aeque difficultatibus implicant.

79. Sint enim ultimae particulae omnis extenſionis expertes, ita ut partibus prorſus careant; qua explicatione quidem ideam entium ſimplicium optime tuentur. At, quemadmodum corpus ex finito huiuſmodi particularum numero conſtare queat, concipi nullo modo poteſt. Ponamus pedem cubicum materiae ex mille huiuſmodi entibus ſimplicibus eſſe compoſitum, huncque actu in mille partes ſecari; quae ſi ſint aequales, erunt digiti cubici: ſin autem ſint inaequales, aliae erunt maiores aliae minores. Unus igitur digitus cubicus foret ens ſimplex, ſicque maxima reſultaret contradictio; niſi forte in digito cubico ineſſe tantum unum ens ſimplex, reliquumque ſpatium vacuum eſſe dicere velint: at vero hoc modo continuitatem corporum tollerent, praeterquam quod iſti Philoſophi vacuum plane ex mundo profligant.

Quodſi

Quodſi obiiciant numerum entium ſimplicium, quae pedem cubicum materiae conſtituunt, millenario longe eſſe maiorem, nihil omnino lucrantur: incommodum enim, quod ex numero millenario ſequitur, ex quovis alio numero quantumvis magno aeque manat. Hanc difficultatem Acutiſſimus LEIBNIZIUS, primus monadum inventor, probe perſpexit, dum materiam abſolute in infinitum diviſibilem eſſe ſtatuit. Neque ergo ante ad monades pervenire licet, quam corpus actu in infinitum ſit diviſum. Hoc ipſo autem exiſtentiam entium ſimplicium, ex quibus corpora conſtent, penitus tollit: nam qui negat corpora ex entibus ſimplicibus eſſe compoſita, & ille qui ſtatuit corpora in infinitum eſſe diviſibilia, in eadem prorſus ſunt ſententia.

80. Neque magis autem ſibi conſtant, ſi dicunt ultimas corporum particulas extenſas quidem eſſe, ſed ob ſummam duritiem in partes divelli non poſſe. Cum primum enim in ultimis particulis extenſionem admittunt, eas ex partibus compoſitas eſſe ſtatuunt, quae, utrum revera a ſe invicem ſeparari queant nec ne, parum refert; etiamſi nullam cauſam aſſignare poſſint, unde tanta durities ſit orta. Nunc autem plerique, qui diviſibilitatem materiae in infinitum negant; hoc poſterius incommodum ſatis ſenſiſſe videntur, quia priori ideae partium ultimarum potiſſimum inhaerent; haſque difficultates aliter diluere non poſſunt, niſi aliquot leviuſculis metaphyſicis diſtinctionibus, quae maximam partem eo tendunt, ut ne conſequentiis, quae ſecundum mathematica principia formantur, fidamus; neque dimenſiones in partibus ſimplicibus adhiberi oportere regerunt. At primum demonſtrare debuiſſent, iſtas ſuas partes ultimas, quarum determinatus numerus corpus conſtituat, extenſas prorſus non eſſe.

81. Cum igitur ex hoc labyrintho exitum nullum invenire, neque obiectionibus debito modo occurrere queant, ad diſtinctiones confugiunt, reſpondentes has obiectiones a

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ſenſibus atque imaginatione ſuppeditari, in hoc autem negotio ſolum intellectum purum adhiberi oportere; ſenſus autem ac ratiocinia inde pendentia ſaepiſſime fallere. Intellectus ſcilicet purus agnoſcet fieri poſſe, ut pars milleſima pedis cubici materiae omni extenſione careat, quod imaginationi abſurdum videtur. Tum vero, quod ſenſus ſaepenumero fallant, res vera quidem eſt, at nemini minus quam mathematicis opponi poteſt. Matheſis enim nos imprimis a fallacia ſenſuum defendit, atque docet obiecta, quae ſenſibus percipiuntur, aliter revera eſſe comparata, aliter vero apparere: haecque ſcientia tutiſſima tradit praecepta, quae qui ſequuntur, ab illuſione ſenſuum immunes ſunt. Huiuſmodi ergo reſponſionibus, tantum abeſt, ut Metaphyſici ſuam doctrinam tueantur, ut eam potius magis ſuſpectam efficiant.

82. Verum ut ad propoſitum revertamur, etiamſi quis neget in mundo numerum infinitum revera exiſtere; tamen in ſpeculationibus mathematicis ſaepiſſime occurrunt queſtiones, ad quas, niſi numerus infinitus admittatur, reſponderi non poſſet. Sic, ſi quaeratur ſumma omnium numerorum, qui hanc ſeriem $1 + 2 + 3 + 4 + 5 +$ &c. conſtituunt; quia iſti numeri ſine fine progrediuntur, atque creſcunt, eorum omnium ſumma certe finita eſſe non poterit: quo ipſo efficitur, eam eſſe infinitam. Hinc, quae quantitas tanta eſt, ut omni quantitate finita ſit maior, ea non infinita eſſe nequit. Ad huiuſmodi quantitatem deſignandam Mathematici utuntur hoc ſigno ∞, quo denotatur quantitas omni quantitate finita, ſeu aſſignabili, maior. Sic cum Parabola ita definiri queat, ut dicatur eſſe Ellipſis infinite longa, recte affirmare poterimus axem Parabolae eſſe Lineam rectam infinitam.

83. Haec autem infiniti doctrina magis illuſtrabitur, ſi quid ſit infinite parvum Mathematicorum expoſuerimus. Nullum autem eſt dubium, quin omnis quantitas eouſque diminui queat, quoad penitus evaneſcat, atque in nihilum abeat. Sed quantitas infinite parva nil aliud eſt niſi quantitas

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evaneſcens, ideoque revera erit $=0$. Conſentit quoque ea infinite parvorum definitio, qua dicuntur omni quantitate aſſignabili minora: ſi enim quantitas tam fuerit parva, ut omni quantitate aſſignabili ſit minor, ea certe non poterit non eſſe nulla; namque niſi eſſet $=0$, quantitas aſſignari poſſet ipſi aequalis, quod eſt contra hypotheſin. Quaerenti ergo, quid ſit quantitas infinite parva in Matheſi, reſpondemus eam eſſe revera $=0$: neque ergo in hac idea tanta Myſteria latent, quanta vulgo putantur, & quae pluribus calculum infinite parvorum admodum ſuſpectum reddiderunt. Interim tamen dubia, ſi quae ſupererunt, in ſequentibus, ubi hunc calculum ſumus tradituri, funditus tollentur.

84. Cum igitur oſtenderimus, quantitatem infinite parvam revera eſſe cyphram, primum occurrendum eſt obiectioni, cur quantitates infinite parvas non perpetuo eodem charactere o deſignemus, ſed peculiares notas ad eas deſignandas adhibeamus. Quia enim omnia nihila ſunt inter ſe aequalia, ſuperfluum videtur variis ſignis ea denotare. Verum quamquam duae quaevis cyphrae ita inter ſe ſunt aequales, ut earum differentia ſit nihil: tamen, cum duo ſint modi comparationis, alter arithmeticus, alter geometricus; quorum illo differentiam, hoc vero quotum ex quantitatibus comparandis ortum ſpectamus; ratio quidem arithmetica inter binas quaſque cyphras eſt aequalitatis, non vero ratio geometrica. Facillime hoc perſpicietur ex hac proportione geometrica $2:1=0:0$, in qua terminus quartus eſt $=0$, uti tertius. Ex natura autem proportionis, cum terminus primus duplo ſit maior quam ſecundus, neceſſe eſt, ut & tertius duplo maior ſit quam quartus.

85. Haec autem etiam in vulgari Arithmetica ſunt planiſſima: cuilibet enim notum eſt, cyphram per quemvis numerum multiplicatam dare cyphram, eſſeque $n.0=0$, ſicque fore $n:1=0:0$. Unde patet fieri poſſe, ut duae cyphrae quamcumque inter ſe rationem geometricam teneant,

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vero ratio aequalitatis inde patet, quod ſit $\frac{a \pm n\,dx}{a} = 1$.
Hinc ſequitur canon ille maxime receptus, quod *infinite parva prae finitis evaneſcant, atque adeo horum reſpectu reiici queant*. Quare illa obiectio, qua Analyſis infinitorum rigorem geometricum negligere arguitur, ſponte cadit, cum nil aliud reiiciatur, niſi quod revera ſit nihil. Ac propterea iure affirmare licet, in hac ſublimiori ſcientia rigorem geometricum ſummum, qui in Veterum libris deprehenditur, aeque diligenter obſervari.

88. Quoniam quantitas infinite parva dx revera eſt $= 0$, eius quoque quadratum dx^2, cubus dx^3, & quaevis alia poteſtas affirmativum habens exponentem erit $= 0$, ideoque aeque prae quantitatibus finitis evaneſcent. At vero etiam quantitas infinite parva dx^2 prae ipſa dx evaneſcit; erit enim $dx \pm dx^2$ ad dx in ratione aequalitatis, ſive comparatio arithmetice ſive geometrice inſtituatur. De priori quidem dubium eſt nullum, at geometrice comparando erit

$$dx \pm dx^2 : dx = \frac{dx \pm dx^2}{dx} = 1 \pm dx = 1.$$

Pari modo erit $dx \pm dx^3 = dx$, & generaliter $dx \pm dx^{n+1} = dx$, dummodo ſit n numerus nihilo maior: erit enim ratio geometrica $dx \pm dx^{n+1} : dx = 1 \pm dx^n$; ideoque, ob $dx^n = 0$, ratio aequalitatis. Si igitur uti in poteſtatibus ſit, vocetur dx infinite parvum primi ordinis, dx^2 ſecundi ordinis, dx^3 tertii ordinis & ita porro, manifeſtum eſt prae infinite parvis primi ordinis, evaneſcere infinite parva altiorum ordinum.

89. Simili modo oſtendetur infinite parva tertii ac ſuperiorum ordinum evaneſcere prae infinite parvis ordinis ſecundi; atque in genere infinite parva cuiuſque ordinis ſuperioris evaneſcere prae infinite parvis ordinis inferioris. Ita ſi m fuerit numerus minor quam n, erit $a\,dx^m + b\,dx^n = a\,dx^m$,

quia dx^n evaneſcit prae dx^m, uti oſtendimus. Hocque etiam in exponentibus fractis habet locum; ita dx evaneſcet prae $\sqrt{dx}$ ſeu $dx^{\frac{1}{2}}$, eritque $a\sqrt{dx}+bdx=a\sqrt{dx}$. Quodſi autem exponens ipſius dx ſit $=0$, erit $dx^0=1$, quamvis ſit $dx=0$; hinc poteſtas dx^n, cum fiat $=1$, ſi ſit $n=0$, ex finita ſtatim fit quantitas infinite parva, atque exponens n nihilo fit maior. Hinc ergo infiniti ordines infinite parvorum exiſtunt, quae etſi omnia ſint $=0$, tamen inter ſe probe diſtingui debent, ſi ad earum relationem mutuam, quae per rationem geometricam explicatur, attendamus.

90. Stabilita notione infinite parvorum facilius indolem infinitorum ſeu infinite magnorum exponere poterimus. Notum eſt valorem fractionis $\frac{1}{z}$ eo maiorem evadere, quo magis diminuatur denominator z; quare ſi z fiat quantitas omni aſſignabili quantitate minor, ſeu infinite parva, neceſſe eſt ut valor fractionis $\frac{1}{z}$ fiat omni aſſignabili quantitate maior, ideoque infinitus. Quamobrem ſi unitas ſeu quaevis alia quantitas finita dividatur per infinite parvum ſeu 0, quotus erit infinite magnus, ideoque quantitas infinita. Cum igitur hoc ſignum ∞ denotet quantitatem infinite magnam, iſta habebitur aequatio $\frac{a}{dx}=\infty$; cuius veritas quoque hinc patet, quod ſit invertendo $\frac{a}{\infty}=dx=0$. Namque quo maior ſtatuitur fractionis $\frac{a}{z}$ denominator z, eo minor fit fractionis valor, atque ſi z fiat quantitas infinite magna ſeu $z=\infty$, neceſſe eſt, ut fractionis valor $\frac{a}{\infty}$ fiat infinite parvus

91. Qui utrumvis horum ratiociniorum negaverit eum

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in maxima incommoda prolabi, atque adeo certiſſima Analyſeos fundamenta evertere neceſſe eſt. Qui enim ſtatuit valorem fractionis $\frac{a}{0}$ eſſe finitum uti b, utrinque per denominatorem multiplicando prodiret $a = 0 . b$, atque ideo quantitas finita b per nihil o multiplicata praeberet quantitatem finitam a, quod eſſet abſurdum. Multo minus valor ille b fractionis $\frac{a}{0}$ poterit eſſe $= 0$: nam o per o multiplicata quantitatem a producere nullo modo poterit. In idem abſurdum incidit, qui negat eſſe $\frac{a}{\infty} = 0$, ei enim dicendum erit eſſe $\frac{a}{\infty} =$ quantitati finitae b: quare cum ex aequatione $\frac{a}{\infty} = b$ legitime ſequatur haec $\infty = \frac{a}{b}$, foret valor fractionis $\frac{a}{b}$, cuius numerator ac denominator ſunt quantitates finitae, infinite magnus, quod perinde foret abſurdum. Neque vero etiam valores fractionum $\frac{a}{0}$ & $\frac{a}{\infty}$ imaginarii ſtatui poſſunt; propterea quod valor fractionis, cuius numerator eſt finitus denominator vero imaginarius, neque infinite magnus neque infinite parvus eſſe poteſt.

92. Quantitas ergo infinite magna, ad quam nos haec conſideratio perduxit, & quae ſola in Analyſi infinitorum locum habet, commodiſſime definitur dicendo, quantitatem infinite magnam eſſe quotum, qui ex diviſione quantitatis finitae per infinite parvam oritur. Viciſſim ergo erit quantitas infinite parva quotus, qui oritur ex diviſione quantitatis finitae per infinite magnam. Quare, cum eiuſmodi proportio geometrica ſubſiſtat, ut ſit quantitas infinite parva ad finitam, ita finita ad infinite magnam; uti quantitas infinita infinities maior eſt quam finita, ita quantitas finita infinities

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maior erit quam infinite parva. Huiuſmodi igitur locutiones, quibus plures offenduntur, non ſunt improbandae, cum certiſſimis innitantur principiis. Deinde etiam ex aequatione $\frac{a}{0} = \infty$ ſequitur fieri poſſe, ut nihil per quantitatem infinite magnam multiplicatum producat quantitatem finitam, quod alienum videri poſſet, niſi planiſſime per legitimam conſequentiam eſſet deductum.

93. Quoniam inter infinite parva, ſi ſecundum rationem geometricam inter ſe comparantur, maximum deprehenditur diſcrimen, ita quoque inter quantitates infinite magnas multo maior differentia intercedit, cum non ſolum geometrice ſed etiam arithmetice comparatae diſcrepent. Ponatur quantitas illa infinita, quae ex diviſione quantitatis finitae a per infinitae parvam dx oritur, $= A$, ita ut ſit $\frac{a}{dx} = A$: erit utique $\frac{2a}{dx} = 2A$ & $\frac{na}{dx} = nA$; cum igitur & nA ſit quantitas infinita, ſequitur inter quantitates infinite magnas rationem quamcunque locum habere poſſe. Hincque, ſi quantitas infinita per numerum finitum ſive multiplicetur, ſive dividatur, prodibit quantitas infinita. Neque ergo de quantitatibus infinitis negari poteſt, eas ulterius augeri poſſe. Facile autem perſpicitur, ſi ratio geometrica, quam duae quantitates infinitae inter ſe tenent, non fuerit aequalitatis, multo minus earum rationem arithmeticam aequalitatis eſſe poſſe, cum potius earum differentia ſemper ſit infinite magna.

94. Quantumvis autem nonnullis idea infiniti, qua in Matheſi utimur, ſuſpecta videatur, qui hanc ob cauſam Analyſin infinitorum profligandam arbitrantur; tamen hac idea ne in partibus quidem Matheſeos trivialibus carere poſſumus. In Arithmetica enim, ubi doctrina logarithmorum tradi ſolet, logarithmus cyphrae & negativus & infinite magnus

gnus ſtatuitur, neque quiſquam eſt tam mente captus, ut hunc logarithmum vel finitum vel adeo nihilo aequalem dicere audeat. In Geometria autem & Trigonometria hoc clarius apparet; quis enim unquam negabit tangentem ſecantemve anguli recti non eſſe infinite magnam? & cum rectangulum ex tangente in cotangentem ſit radii quadrato aequale, cotangens autem anguli recti ſit $=0$; in Geometria adeo concedi debet, productum ex nihilo & infinito eſſe poſſe finitum.

95. Cum ſit $\frac{a}{dx}$ quantitas infinita A, patet hanc quantitatem $\frac{A}{dx}$ fore quantitatem infinities maiorem, quam A: eſt enim $\frac{a}{dx} : \frac{A}{dx} = a : A$, hoc eſt ut numerus finitus ad infinite magnum. Dantur ergo inter quantitates infinite magnas eiuſmodi relationes, ut aliae aliis infinities maiores eſſe queant. Sic $\frac{a}{dx^2}$ erit quantitas infinita infinities maior quam $\frac{a}{dx}$; poſito enim $\frac{a}{dx} = A$ erit $\frac{a}{dx^2} = \frac{A}{dx}$. Simili modo erit $\frac{a}{dx^3}$ quantitas infinita infinities maior quam $\frac{a}{dx^2}$, ideoque infinities infinities maior quam $\frac{a}{dx}$. Dantur ergo infiniti gradus infinitorum, quorum quiſque infinities maior eſt quam praecedentes: atque adeo ſi numerus m vel tantillum maior ſit quam n erit $\frac{a}{dx^m}$ quantitas infinita infinities maior quam quantitas infinita $\frac{a}{dx^n}$.

96. Quemadmodum in quantitatibus infinite parvis dantur rationes geometricae inaequales, cum tamen rationes arithmeticae omnes ſint aequales: ita in quantitatibus infinite magnis dantur rationes geometricae aequales, cum tamen arithmeticae ſint quantumvis inaequales. Si enim a & b denotent quantitates finitas, hae duae quantitates infinitae $\frac{a}{dx} + b$ & $\frac{a}{dx}$ rationem geometricam habent aequalitatis; erit enim quotus ex earum diviſione ortus $= 1 + \frac{bdx}{a} = 1$ ob $dx = 0$: interim tamen, ſi arithmetice compararentur, ob differentiam $= b$, ratio erit inaequalitatis. Simili modo $\frac{a}{dx^2} + \frac{a}{dx}$ ad $\frac{a}{dx^2}$ rationem geometricam habet aequalitatis, exponens enim rationis eſt $= 1 + dx = 1$; verum tamen differentia eſt $\frac{a}{dx}$ ideoque infinita. Hinc ſi ad rationem geometricam ſpectemus, infinite magna inferiorum graduum prae infinite magnis ſuperiorum graduum evaneſcunt.

97. His de gradibus infinitorum praemoniiis, mox apparebit fieri poſſe, ut productum ex quantitate infinite magna in infinite parvam non ſolum quantitatem finitam producat, quod ſupra eveniſſe vidimus; ſed etiam huiuſmodi productum eſſe poterit ſive infinite magnum ſive infinite parvum. Sic quantitas infinita $\frac{a}{dx}$, ſi per infinite parvam dx multiplicetur, dat productum finitum $= a$; ſin autem $\frac{a}{dx}$ multiplicetur per infinite parvum dx^2, vel dx^3, vel aliud ſuperioris ordinis, productum erit vel adx, vel adx^2, vel adx^3 &c. ideoque infinite parvum. Eodem modo intelligetur, ſi quantitas infinita $\frac{a}{dx^2}$ multiplicetur per infinite par-

vam dx, productum fore infinite magnum: atque generatim ſi $\frac{a}{dx^n}$ multiplicetur per bdx^m, productum $abdx^{m-n}$ erit infinite parvum ſi m ſuperat n; finitum ſi m aequat n; & infinite magnum ſi m ſuperatur ab n.

98. Quantitates tam infinite parvae, quam infinite magnae in ſeriebus numerorum ſaepiſſime occurrunt, in quibus cum ſint numeris finitis permixtae, ex iis luculenter patebit, quemadmodum ſecundum leges continuitatis a quantitatibus finitis ad infinite magnas atque infinite parvas tranſitio fiat. Conſideremus primum ſeriem numerorum naturalium, quae ſimul retro continuata erit

&c. $-4-3-2-1+0+1+2+3+4+$ &c.

Numeri ergo continuo decreſcendo praebent tandem o ſeu infinite parvum, unde ulterius continuati negativi evadunt. Quamobrem hinc intelligitur a numeris finitis affirmativis decreſcentibus tranſiri per o ad negativos creſcentes. Sin autem eorum numerorum quadrata ſpectentur, quia omnia ſunt affirmativa

&c. $+16+9+4+1+0+1+4+9+16+$ &c.

erit o quoque tranſitus numerorum affirmativorum decreſcentium ad affirmativos creſcentes; atque ſi ſigna mutentur, erit quoque o tranſitus numerorum negativorum decreſcentium ad negativos creſcentes.

99. Si ſeries conſideretur, cuius terminus generalis eſt $\sqrt{x}$, quae etiam retro continuata erit huiuſmodi

&c. $+\sqrt{-3}+\sqrt{-2}+\sqrt{-1}+0+\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}+$ &c.

ex qua patet o, quoque tanquam limitem conſiderari poſſe, per quem a quantitatibus realibus ad imaginaria tranſeatur. Si iſti termini tanquam applicatae curvarum conſiderentur, perſpicitur, ſi eae fuerint affirmativae atque eouſque decreverint ut tandem evaneſcant, tum eas ulterius continuatas vel fieri negativas, vel iterum affirmativas vel adeo imaginarias.

Idem

Idem eveniet, ſi applicatae primum fuerint negativae; tum enim aeque poſt quam evanuerint, ſi ulterius continuentur, vel affirmativae fient, vel negativae vel imaginariae; quorum phaenomenorum plurima exempla praebet doctrina de lineis curvis in libro praecedente tractata.

100. Eodem modo in ſeriebus occurrunt ſaepe termini infiniti: ſic in ſerie harmonica, cuius terminus generalis eſt $\frac{1}{x}$, indici $x = 0$ reſpondebit terminus infinite magnus $\frac{1}{0}$; totaque ſeries ita ſe habebit:

$$\&c. - \tfrac{1}{4} - \tfrac{1}{3} - \tfrac{1}{2} - \tfrac{1}{1} + \tfrac{1}{0} + \tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} + \&c.$$

A dextra ergo ad ſiniſtram progrediendo termini creſcunt, ita ut $\frac{1}{0}$ iam ſit infinite magnus, quem cum tranſierint, fient negativi decreſcentes. Hinc quantitas infinite magna ſpectari poteſt tanquam limes, per quem numeri affirmativi progreſſi fiunt negativi, & viciſſim: unde pluribus viſum eſt, numeros negativos conſiderari poſſe, tanquam infinito maiores, propterea quod in hac ſerie termini continuo creſcentes, poſtquam infinitum attigerint, abeant in negativos. At vero ſi ad ſeriem, cuius terminus generalis eſt $\frac{1}{xx}$, attendamus, poſt tranſitum per infinitum rurſus prodeunt termini affirmativi.

$$\&c. \tfrac{1}{9} + \tfrac{1}{4} + \tfrac{1}{1} + \tfrac{1}{0} + \tfrac{1}{1} + \tfrac{1}{4} + \tfrac{1}{9} + \&c.$$

quos tamen nemo infinito maiores dixerit.

101. Saepenumero quoque in ſeriebus terminus infinitus conſtituit limitem, terminos reales ab imaginariis ſegregantem, uti fit in ſerie hac, cuius terminus generalis eſt $\frac{1}{\sqrt{x}}$

$$\&c. + \frac{1}{\sqrt{-3}} + \frac{1}{\sqrt{-2}} + \frac{1}{\sqrt{-1}} + \frac{1}{0} + \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \&c.$$

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neque tamen hinc ſequitur, imaginaria eſſe infinito maiora: quoniam ex ſerie ante allata

$$\&c. +\sqrt{-3}+\sqrt{-2}+\sqrt{-1}+0+\sqrt{1}+\sqrt{2}+\sqrt{3}+ \&c.$$

aeque ſequeretur, imaginaria eſſe nihilo minora. Deinde vero etiam a terminis realibus tranſitus ad imaginarios exhiberi poteſt, quorum limes neque ſit 0 neque ∞, uti fit ſi terminus generalis fuerit $1+\sqrt{x}$. His autem caſibus, cum ob irrationalitatem quilibet terminus geminum habeat valorem, in limite inter realia & imaginaria ſemper bini illi valores fiunt inter ſe aequales. At quoties termini, qui ante erant affirmativi, abeunt in negativos, tranſitus ſemper fit per limitem vel infinite parvum, vel infinite magnum, quae omnia ex lege continuitatis, quam in lineis curvis deprehendimus, clarius elucent.

102. Ex ſummatione quoque ſerierum in infinitum excurrentium plura hic afferri poſſunt, quae cum ad hanc infiniti doctrinam magis illuſtrandam, tum vero ad plura dubia, quae in hoc negotio ſuboriri ſolent, delenda inſerviunt. Ac primo quidem, ſi ſeries conſtet ex terminis aequalibus, ut

$$1+1+1+1+1+1+ \&c.$$

eaque ſine fine, hoc eſt in infinitum continuetur, nullum certe eſt dubium, quin omnium horum terminorum ſumma maior ſit omni numero aſſignabili; eaque propterea infinita ſit neceſſe eſt. Hoc quoque confirmat eius origo, dum oritur ex evolutione fractionis

$$\frac{1}{1-x} = 1+x+x^2+x^3+ \&c.$$

ponendo $x=1$; erit ergo

$$\frac{1}{1-1} = 1+1+1+1+ \&c.$$

ideoque ſumma $= \frac{1}{1-1} = \frac{1}{0} =$ infinito.

103. Quamvis autem hic nullum dubium naſci queat, cum idem numerus finitus infinities ſumtus in infinitum abire debeat; tamen ipſa origo ex ſerie generali

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \&c.$$

graviſſima incommoda afferre videtur: ſi enim pro x ſucceſſive ponantur numeri 1, 2, 3, &c. ſequentes ſeries cum ſuis ſummis prodibunt.

$$A \ldots 1 + 1 + 1 + 1 + 1 + \&c. = \frac{1}{1-1} = \text{infinito}$$

$$B \ldots 1 + 2 + 4 + 8 + 16 + \&c. = \frac{1}{1-2} = -1$$

$$C \ldots 1 + 3 + 9 + 27 + 81 + \&c. = \frac{1}{1-3} = -\frac{1}{2}$$

$$D \ldots 1 + 4 + 16 + 64 + 256 + \&c. = \frac{1}{1-4} = -\frac{1}{3}$$

&c.

Cum igitur ſeries B ſingulos terminos praeter primum habeat maiores, quam ſeries A, ſumma ſeriei B neceſſario multo maior eſſe deberet, quam ſumma ſeriei A: interim tamen iſte calculus oſtendit ſeriei A ſummam infinitam, ſeriei B vero ſummam negativam, hoc eſt nihilo minorem, quod concipi non poteſt. Multo minus cum ſolitis ideis conciliari poteſt, quemadmodum huius & ſequentium ſerierum C, D, &c. ſummae fiant negativae, cum tamen omnes termini ſint affirmativi.

104. Ob hanc rationem opinio ſupra allata multis probabilis videri ſolet, quantitates ſcilicet negativas quandoque conſiderari poſſe tanquam infinito maiores ſeu plus quam infinitas; & cum etiam numeros ultra nihil diminuendo perveniatur ad negativos, diſcrimen ſtatuunt inter numeros negativos huiuſmodi -1, -2, -3, &c. & huiuſmodi

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$\frac{+1}{-1}, \frac{+2}{-1}, \frac{+3}{-1}$, &c. illos nihilo minores, hos vero infinito maiores dicendo. Verumtamen hoc pacto difficultatem non tollunt, quam ſuggerit haec ſeries

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \&c. = \frac{1}{(1-x)^2}$$

unde oriuntur ſequentes ſeries:

$$A \ldots 1 + 2 + 3 + 4 + 5 + \&c. = \frac{1}{(1-1)^2} = \frac{1}{0} = \text{infinito}$$

$$B \ldots 1 + 4 + 12 + 32 + 80 + \&c. = \frac{1}{(1-2)^2} = 1$$

ubi cum ſinguli termini ſeriei B ſint maiores, quam ſinguli termini ſeriei A, primis ſolis exceptis, quemadmodum ſumma ſeriei A ſit infinita, ſeriei B vero ſumma aequalis 1, hoc eſt ſoli termino primo, ex illo principio explicari omnino nequit.

105. Quoniam autem ſi vellemus negare eſſe $-1 = \frac{+1}{-1}$, & $\frac{+a}{-b} = \frac{-a}{+b}$, firmiſſima Analyſeos fundamenta collaberentur, illa ante commemorata explicatio prorſus admitti non poteſt. Quin potius negare debebimus, illas, quas formulae generales ſuppeditaverant, ſummas eſſe veras. Cum enim hae ſeries ex continua diviſione oriantur, dum reſiduum continuo ulterius dividitur: reſiduum autem perpetuo fiat maius, quo longius progrediamur, id nunquam negligere poterimus; atque minime reſiduum ultimum, hoc eſt quod in diviſione infiniteſima remanet, omitti poteſt, quippe quod ſit infinite magnum. Quia autem hoc in ſuperioribus ſeriebus non obſervatur, dum nullius reſidui ratio habetur, mirum non eſt, eas ſummationes ad abſurdum deducere. Haecque reſponſio, uti eſt ex ipſa ſerierum geneſi petita, ita quoque eſt veriſſima, atque omnem dubitationem tollit.

K 2 106.

106. Quo hoc clarius appareat, contemplemur evolutionem fractionis $\frac{1}{1-x}$, uti in terminis primum finitis tantum abſolvitur. Erit ergo

$$\frac{1}{1-x} = 1 + \frac{x}{1-x}$$

$$\frac{1}{1-x} = 1 + x + \frac{x^2}{1-x}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \frac{x^3}{1-x}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \frac{x^4}{1-x}$$

&c.

qui ergo dicere vellet huius ſeriei finitae $1 + x + x^2 + x^3$ ſummam eſſe $\frac{1}{1-x}$, is erraret a vero quantitate $\frac{x^4}{1-x}$: & qui ſummam huius ſeriei

$$1 + x + x^2 + x^3 + \ldots\ldots + x^{1000}$$

ſtatuere vellet $= \frac{1}{1-x}$, is erraret quantitate $\frac{x^{1001}}{1-x}$ qui error ſi x ſit numerus unitate maior, foret maximus.

107. Ex his perſpicuum eſt eum, qui eiuſdem ſeriei in infinitum continuatae ſeu huius:

$$1 + x + x^2 + x^3 + \ldots\ldots + x^{\infty}$$

ſummam ſtatuere velit $= \frac{1}{1-x}$, a veritate eſſe aberraturum quantitate $\frac{x^{\infty+1}}{1-x}$; quae ſi ſit $x > 1$ utique erit infinite magna. Simul vero hinc ratio patet, cur ſeriei in infinitum continuatae

$$1 + x + x^2 + x^3 + x^4 + \text{\&c.}$$

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ſumma revera fit $= \frac{1}{1-x}$, ſi fuerit x fractio unitate minor, tum enim error $\frac{x^{\infty+1}}{1-x}$ fit infinite parvus, ideoque nullus; cuius propterea ratio tuto poteſt negligi. Sic poſito $x = \frac{1}{2}$, erit revera

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \&c. = \frac{1}{1-\frac{1}{2}} = 2,$$

ſimiliterque reliquarum ſerierum, ſi x ſit fractio unitate minor, ſumma vera hoc modo indicatur.

108. Haec eadem reſponſio valet de ſummis ſerierum divergentium, in quibus ſigna $+$ & $-$ alternantur, quae vulgo ex eadem formula exhiberi ſolent, ponendo pro x numeros negativos. Cum enim ſit:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \&c.$$

niſi ultimi reſidui ratio habeatur, foret:

A . . . $1 - 1 + 1 - 1 + 1 - 1 + \&c. = \frac{1}{2}$
B . . . $1 - 2 + 4 - 8 + 16 - 32 + \&c. = \frac{1}{3}$
C . . . $1 - 3 + 9 - 27 + 81 - 243 + \&c. = \frac{1}{4}$
&c.

Patet autem ſeriei ſecundae B ſummam ideo non poſſe eſſe $= \frac{1}{3}$, cum quo plures termini actu ſummentur, aggregata eo magis ab $\frac{1}{3}$ recedant. Perpetuo autem cuiuſque ſeriei ſumma debet eſſe limes, ad quem eo propius perveniatur, quo plures termini actu addantur.

109. Ex his quidam concluſerunt huiuſmodi ſeries, quae vocantur divergentes, prorſus nullas habere ſummas fixas; propterea quod colligendis actu terminis ad nullum limitem fiat appropinquatio, qui pro ſumma ſeriei in infinitum continuatae haberi poſſet: quae ſententia, cum iſtae ſummae iam ob neglecta ultima reſidua erroneae ſint oſten-

ſae,

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ſae, veritati maxime eſt conſentanea. Interim tamen contra eam ſummo iure obiici poteſt, has memoratas ſummas, quantumvis a veritate abhorrere videantur, tamen nunquam in errores inducere; quin potius iis admiſſis plurima praeclara eſſe eruta, quibus ſi iſtas ſummationes prorſus reiicere vellemus, carendum eſſet. Neque vero hae ſummae, ſi eſſent falſae, perpetuo ad veritatem nos ducere poſſent; quin potius, cum non parum ſed infinite a veritate diſcrepent, nos quoque in infinitum a vero ſeducere deberent. Quod tamen cum non eveniat, difficillimus nobis reſtat nodus ſolvendus.

110. Dico igitur in voce *ſummae* latere totam difficultatem; ſi enim *ſumma* ſeriei, ut vulgo uſus fert, ſumatur pro aggregato omnium eius terminorum actu collectorum, tum dubium eſt nullum, quin earum tantum ſerierum in infinitum excurrentium ſummae exhiberi queant, quae ſint convergentes, atque continuo propius ad certum ſtatumque valorem deducant, quo plures termini actu colligantur. Series autem divergentes, quarum termini non decreſcunt, ſive ſigna + & — alternentur ſive ſecus, prorſus nullas habebunt ſummas fixas; ſi quidem vox ſummae hoc ſenſu pro aggregato omnium terminorum accipiatur. At vero in iis caſibus, quorum meminimus, quibus ex iſtiuſmodi ſummis erroneis veritas tamen elicitur; id non fit, quatenus expreſſio finita, verbi gratia $\frac{1}{1-x}$, eſt ſumma ſeriei $1+x+x^2+x^3+$ &c. ſed quatenus ea expreſſio evoluta hanc ſeriem praebet; ſicque in hoc negotio nomen ſummae prorſus omitti poſſet.

111. Haec igitur incommoda, haſque apparentes contradictiones penitus evitabimus, ſi voci *ſummae* aliam notionem, atque vulgo fieri ſolet, tribuamus. Dicamus ergo ſeriei cuiuſque infinitae *ſummam* eſſe expreſſionem finitam, ex cuius evolutione illa ſeries naſcatur. Hocque ſenſu ſeriei in-

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infinitae $1 + x + x^2 + x^3 +$ &c. ſumma revera erit $= \frac{1}{1-x}$, quia illa ſeries ex huius fractionis evolutione oritur: quicunque numerus loco x ſubſtituatur. Hoc pacto, ſi ſeries ſuerit convergens, iſta nova vocis ſummae definitio, cum conſueta congruet; & quia divergentes nullas habent ſummas proprie ſic dictas, hinc nullum incommodum ex nova hac appellatione orietur. Denique ope huius definitionis utilitatem ſerierum divergentium tueri, atque ab omnibus iniuriis vindicare poterimus.

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CAPUT IV.
DE DIFFERENTIALIUM CUIUSQUE ORDINIS NATURA

112.

In capite primo vidimus, ſi quantitas variabilis x accipiat augmentum $=\omega$, tum cuiuſvis functionis ipſius x augmentum inde oriundum tali forma exprimi $P\omega + Q\omega^2 + R\omega^3 +$ &c. ſive haec expreſſio ſit finita ſive in infinitum excurrat. Functio ergo y, ſi in ea loco x ſcribatur $x+\omega$, valorem ſequentem induet:

$$y^{I} = y + P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \&c.$$

a quo, ſi valor prior y ſubtrahatur, remanebit differentia functionis y, quae ita exprimetur

$$\Delta y = P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \&c.$$

atque cum valor ipſius x ſequens ſit $x^{I} = x + \omega$, erit differentia ipſius x, nempe $\Delta x = \omega$. Litterae autem P, Q, R, &c. denotant functiones ipſius x pendentes ab y, quas capite primo invenire docuimus.

113. Hinc ergo quocunque augmento ω augeatur quantitas variabilis x, ſimul definiri poterit augmentum, quod cuique ipſius x functioni y accedit; dummodo pro quovis ipſius y valore functiones P, Q, R, S, &c. definire valeamus. In hoc autem capite, atque in univerſa Analyſi infinitorum augmentum illud ω, quo quantitatem variabilem x creſcere ſumſimus, ſtatuemus infinite parvum, atque adeo evaneſcens, ſeu $= 0$. Unde manifeſtum eſt, incrementum ſeu differentiam functionis y quoque fore infinite parvam. Cum autem in hac hypotheſi ſinguli termini expreſſionis

$$P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \&c.$$

prae antecedentibus evaneſcant, (88. & ſeqq.), ſolus primus

$P\omega$

$P\omega$ remanebit, eritque propterea hoc caſu, quo ω eſt infinite parvum, differentia ipſius y nempe $\Delta y = P\omega$.

114. Erit ergo Analyſis infinitorum, quam hic tractare coepimus, nil aliud, niſi caſus particularis methodi differentiarum in capite primo expoſitae, qui oritur, dum differentiae, quae ante finitae erant aſſumtae, ſtatuantur infinite parvae. Quo igitur iſte caſus, quo univerſa Analyſis infinitorum continetur, a methodo differentiarum diſtinguatur, cum peculiaribus nominibus, tum etiam ſignis ad differentias iſtas infinite parvas denotandas uti conveniet. Differentias igitur infinite parvas hic cum LEIBNIZIO *differentialia* vocabimus; atque cum differentiarum in primo capite diverſos ordines conſtituiſſemus, ex iis nunc facile quoque intelligetur, quid differentialia prima, ſecunda, tertia, &c. cuiuſque functionis ſignificent. Loco characteris autem Δ, quo ante differentias indicaveramus, nunc utemur charactere d; ita ut dy ſignificet differentiale primum ipſius y; ddy differentiale ſecundum; d^3y tertium & ita porro.

115. Quoniam differentias infinite parvas, quas hic tractamus *differentialia* vocamus, hinc totus calculus, quo differentialia inveſtigantur, atque ad uſum accomodantur, appellari ſolet *Calculus differentialis*. Mathematici Angli, inter quos primum NEWTONUS aeque ac LEIBNIZIUS inter Germanos hanc novam Analyſeos partem excollere coepit, aliis tam nominibus quam ſignis utuntur. Differentias enim infinite parvas, quas nos differentialia vocamus potiſſimum *fluxiones* nominare ſolent, interdum quoque *incrementa*: quae voces utique latino ſermoni magis conveniunt, ita quoque res, quas denotant, ſatis commode exprimunt. Quantitas enim variabilis creſcendo continuo alios atque alios valores recipiens tanquam fluens conſiderari poteſt, hincque vox fluxionis, quae primum a NEWTONO ad celeritatem creſcendi adhibebatur, ad incrementum infinite parvum, quod quantitas quaſi fluendo accipit, deſignandum analogice eſt tranſlata.

116. Quamvis autem circa vocum uſum atque definitionem cum Anglis diſceptare abſonum foret, noſque coram iudice puritatem latinae linguae atque expreſſionum commoditatem ſpectante facile ſuperaremur; tamen nullum eſt dubium, quin Anglis ratione ſignorum palmam praeripiamus. Differentialia enim, quae ipſi fluxiones appellant, punctis, quae litteris ſuperſcribunt, denotare ſolent, ita ut $\dot{y}$ iis ſignificet fluxionem primam ipſius y; $\ddot{y}$ fluxionem ſecundam; $\dddot{y}$ fluxionem tertiam, atque ita porro. Qui notandi modus, uti ab arbitrio pendens, etſi improbari nequit, ſi punctorum numerus fuerit parvus, ut numerando facile percipi queat, tamen ſi plura puncta inſcribi debeant, maximam confuſionem plurimaque incommoda affert. Differentiale enim ſeu fluxio decima perquam incommode hoc modo $\overset{\vdots\vdots}{y}$ repraeſentatur, cum noſtro ſignandi modo $d^{10}y$ facillime comprehendatur. Oriuntur autem caſus, quibus multo adhuc ſuperiores differentialium ordines atque adeo indefiniti exprimi debent, ad quos Anglorum modus prorſus ſit ineptus.

117. Noſtris igitur tam nominibus quam ſignis utemur, quippe quorum illa in noſtris regionibus iam ſunt uſu recepta atque pleriſque familiaria, haec vero commodiora. Interim tamen non abs re erat, Anglorum denominationes & ſignationes hic commemorare, ut qui eorum libros evolvunt, eos quoque intelligere queant. Neque enim Angli ſuo mori tam pertinaciter adhaerent, ut quae noſtro more ſunt ſcripta, prorſus repudient, nec legere dignentur. Nos quidem ipſorum opera maxima cum aviditate perlegimus, ex iiſque ſummum fructum percipimus; ſaepenumero vero etiam animadvertimus, ipſos noſtratium ſcripta non ſine utilitate legiſſe. Quamobrem etſi idem ubique atque aequabilis modus cogitata ſua exprimendi maxime eſſet optandus, tamen non admodum eſt difficile, ut utrique aſſueſcamus, quantum quidem

dem intelligentia librorum alieno more ſcriptorum poſtulat.

118. Cum igitur littera ω nobis hactenus denotaverit differentiam ſeu incrementum, quo quantitas variabilis x creſcere concipitur, nunc autem ω ſtatuatur infinite parvum, erit ω differentiale ipſius x; & hancobrem recepto ſignandi modo erit $\omega = dx$; atque dx proinde erit differentia infinite parva, qua ipſa x creſcere concipitur. Simili modo differentiale ipſius y ita exprimetur dy; atque ſi y fuerit functio quaecumque ipſius x, differentiale dy denotabit incrementum, quod functio y capit, dum x abit in $x + dx$. Quare ſi in functione y ubique loco x ſubſtituatur $x + dx$, & quantitas reſultans ponatur $= y^{\mathrm{I}}$, erit $dy = y^{\mathrm{I}} - y$, hocque modo differentiale cuiuſque functionis reperietur: quod quidem intelligendum eſt de differentiali primo ſeu primi ordinis; de reliquis enim poſtea videbimus.

119. Probe ergo tenendum eſt litteram d hic non quantitatem denotare, ſed tantum loco ſigni adhiberi, ad vocem *differentialis* exprimendam, eodem modo, quo in doctrina logarithmorum littera l pro ſigno logarithmi, & in Algebra charactere $\surd$ pro ſigno radicis uti conſuevimus. Hinc dy non ſignificat, uti vulgo in Analyſi uſu eſt receptum, productum ex quantitate d in quantitatem y, ſed ita enunciari debet, ut dicatur differentiale ipſius y. Simili modo ſi ſcribatur $d^2 y$, neque binarius exponentem, neque d^2 poteſtatem ipſius d ſignificat, ſed adhibetur tantum ad nomen *differentialis ſecundi* breviter & apte exprimendum. Cum igitur littera d in calculo differentiali non quantitatem, ſed ſignum tantum exhibeat, ad confuſionem vitandam in calculis, ubi plures quantitates conſtantes occurrunt, littera d ad earum deſignationem uſurpari nequit; perinde atque evitare ſolemus litteram l tanquam quantitatem in calculum inducere, ubi ſimul logarithmi occurrunt. Optandum autem eſſet, ut litterae iſtae d & l per characteres aliquantulum alteratos exprimerentur, ne cum litteris Alphabethi, quibus quantitates de-

ſignari ſolent, confundantur: ſimili ſcilicet modo, quo loco litterae r, qua primum vox radicis indicabatur, nunc character iſte diſtortus $\surd$ in uſum eſt receptus.

120. Quoniam igitur vidimus differentiale primum ipſius y, ſi y fuerit functio quaecunque ipſius x, habiturum eſſe huiuſmodi formam $P\omega$; ob $\omega = dx$, erit $dy = Pdx$. Qualiſcunque ſcilicet fuerit y functio ipſius x, eius differentiale dy exprimetur certa quadam functione ipſius x, pro qua hic ponimus P, per differentiale ipſius x, nempe per dx multiplicata. Etiamſi ergo differentialia ipſarum x & y revera ſint infinite parva, ideoque nihilo aequalia; tamen inter ſe finitam habebunt rationem: erit ſcilicet $dy : dx = P : 1$. Inventa ergo functione iſta P, innoteſcit ratio inter differentiale dx & differentiale dy. Cum igitur calculus differentialis in inventione differentialium conſiſtat, in eo non tam ipſa differentialia, quae ſunt nihilo aequalia ac propterea nullo labore invenirentur, quam eorum ratio mutua geometrica inveſtigatur.

121. Differentialia igitur multo facilius inveniuntur, quam differentiae finitae. Ad differentiam enim finitam Δy, qua functio y creſcit, dum quantitas variabilis x incrementum ω accipit, non ſufficit functionem P noſſe, ſed indagari inſuper oportet functiones Q, R, S, &c. quae in differentiam finitam, quam poſuimus

$$= P\omega + Q\omega^2 + R\omega^3 + \&c.$$

ingrediuntur; ad differentiale ipſius y autem inveniendum ſatis eſt, ſi noverimus ſolam functionem P. Quamobrem ex cognita differentia finita cuiuſque functionis ipſius x, facillime eius differentiale definitur; verum contra ex differentiali eius functionis, nondum erui poteſt eius differentia finita. Interim tamen infra docebitur, quemadmodum ex differentialibus omnium ordinum ſimul cognitis differentia quaevis finita cuiuſque functionis propoſitae inveniri queat. Ceterum ex his

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manifeſtum eſt differentiale primum $dy = Pdx$, praebere terminum primum differentiae finitae, quippe qui eſt $= P\omega$.

122. Si igitur incrementum ω, quod quantitas variabilis x accipere concipitur, fuerit vehementer parvum, ita ut in expreſſione $P\omega + Q\omega^2 + R\omega^3 +$ &c. termini $Q\omega^2$ & $R\omega^3$, multoque magis reliqui, fiant tam parvi, ut in computo, quo ſummus rigor non obſervatur, prae primo $P\omega$ negligi queant; tum cognito differentiali Pdx, ex eo differentia finita vero proxime cognoſcetur, quippe quae erit $= P\omega$: unde in pluribus occaſionibus, quibus calculus ad praxin adhibetur, non parum fructus hauritur. Atque hinc nonuulli arbitrantur, differentialia tanquam incrementa vehementer parva conſiderari poſſe, eaque nihilo revera aequalia eſſe negant, atque tantum indefinite parva ſtatuunt. Haecque idea aliis occaſionem praebuit Analyſin infinitorum accuſandi, quod non veras rerum quantitates eliciat, ſed tantum vero proximas; quae obiectio ſemper aliquam vim retineret, niſi infinite parva prorſus nihilo aequalia ſtatueremus.

123. Qui autem nolunt infinite parva plane in nihilum abire, ii ut vim obiectionis deſtruere videantur, differentialia comparant minimis pulviſculis ratione totius terrae, cuius quantitatem nemo non veram tradidiſſe cenſeretur, qui unico pulviſculo a veritate aberraverit. Talem igitur rationem inter quantitatem finitam & infinite parvam eſſe volunt, qualis eſt inter totam terram minimumque pulviſculum: atque ſi cui hoc diſcrimen adhuc non ſatis magnum videatur, eam rationem millies magiſque adaugent, ut parvitas amplius omnino percipi nequeat. Interim tamen agnoſcere coguntur, ſummum rigorem geometricum aliquantulum infringi; quare quo huic obiectioni occurrant, ad eiuſmodi exempla confugiunt, quorum tam per Geometriam quam per Analyſin infinitorum ſolutiones inveniri poſſunt, ex earumque congruentia bonitatem poſterioris methodi concludunt. Quanquam autem hoc argumentum negotium non conficit, cum ſaepenume-

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ro per erroneas methodos verum elici queat; tamen quia hoc vitio non laborat, potius evincit, eas quantitates, quae in calculo ſint neglectae, non ſolum non incomprehenſibiliter parvas, ſed plane nullas eſſe, uti nos aſſumimus. Ex quo rigori geometrico nullam omnino vim inferimus.

124. Progrediamur ad differentialium ſecundi ordinis naturam explicandam, quae oriuntur ex differentiis ſecundis in capite primo expoſitis, ponendo quantitatem ω infinite parvam $= dx$. Cum igitur ſi ponamus quantitatem variabilem x aequalibus incrementis creſcere, ita ut ſi valor ſecundus x^{I} fuerit $= x + dx$, ſequentes futuri ſint $x^{\text{II}} = x + 2\,dx$; $x^{\text{III}} = x + 3\,dx$ &c. ob differentias primas conſtantes $= dx$, differentiae ſecundae evaneſcent: erit ergo quoque differentiale ſecundum ipſius x nempe $ddx = 0$, atque ab hanc rationem quoque differentialia ulteriora erunt $= 0$, ſcilicet $d^3 x = 0$, $d^4 x = 0$; $d^5 x = 0$; &c. Obiici quidem poteſt, haec differentialia, cum ſint infinite parva, per ſe eſſe $= 0$, neque hoc proprium eſſe eius quantitatis variabilis x, cuius incrementa aequalia concipiantur: at vero hanc evaneſcentiam ita interpretari oportet, ut differentialia ddx, $d^3 x$ &c. non ſolum in ſe ſpectata ſint nulla, ſed etiam ratione poteſtatum ipſius dx, cum quibus alias comparari poſſent, evaneſcere.

125. Quae quo clarius intelligantur, recordandum eſt differentiam ſecundam cuiuſque functionis ipſius x, quae ſit y, huiuſmodi forma exprimi $P\omega^2 + Q\omega^3 + R\omega^4 +$ &c. Quodſi ergo ω ſit infinite parvum, termini $Q\omega^3$, $R\omega^4$ &c. prae primo $P\omega^2$ evaneſcent, unde poſito $\omega = dx$, differentiale ſecundum ipſius y erit $= P\,dx^2$, denotante dx^2 quadratum differentialis dx. Quare etſi differentiale ſecundum ipſius y, nempe ddy per ſe ſit $= 0$, tamen cum ſit $ddy = P\,dx^2$, ad dx^2 habebit rationem finitam uti P ad 1: ſin autem ſit $y = x$, tum ſit $P = 0$, $Q = 0$, $R = 0$, &c. ideoque hoc caſu differentiale ſecundum ipſius x etiam reſpectu dx^2 altiorumque ipſius dx poteſtatum evaneſcit. Hocque modo intelligenda ſunt

ea,

ea, quae ante diximus, eſſe ſcilicet $ddx = 0$, $d^3 x = 0$, &c.

126. Cum differentia ſecunda nil aliud ſit, niſi differentia differentiae primae; differentiale quoque ſecundum ſeu uti ſaepe vocari ſolet, differentio-differentiale nil aliud erit praeter differentiale differentialis primi. Quia deinde quantitas conſtans nulla neque augmenta neque decrementa accipit, nullaſque admittit differentias, quippe quae ſolis quantitatibus variabilibus ſunt propriae, dicimus eodem ſenſu quantitatum conſtantium differentialia omnia cuiuſque ordinis eſſe $= 0$, hoc eſt prae omnibus adeo poteſtatibus ipſius dx evaneſcere. Cum igitur differentiale ipſius dx hoc eſt ddx ſit $= 0$; differentiale dx tanquam quantitas conſtans conſiderari poteſt, & quoties differentiale cuiuſpiam quantitatis dicitur conſtans, toties ea quantitas intelligenda eſt continuo aequalia incrementa accipere. Sumimus hic autem x pro ea quantitate, cuius differentiale ſit conſtans, hicque ſingularum eius functionum variabilitatem, cui earum differentialia ſunt obnoxia, aeſtimabimus.

127. Ponamus differentiale primum ipſius y eſſe $= pdx$; atque ad eius differentiale ſecundum inveniendum, ipſius pdx denuo differentiale quaeri debet. Cum autem dx ſit conſtans, neque varietur etiamſi loco x ſcribatur $x+dx$, tantum opus eſt, ut quantitatis finitae p differentiale quaeratur: ſit igitur $dp = qdx$, quoniam vidimus omnium functionum ipſius x differentialia ad huiuſmodi formam revocari: & cum ſit, uti de differentiis finitis oſtendimus, differentiale ipſius $np = nqdx$, ſi n ſit quantitas conſtans, ponatur dx loco n, eritque differentiale ipſius $pdx = qdx^2$. Hancobrem ſi ſit $dy = pdx$ & $dp = qdx$, erit differentiale ſecundum $ddy = qdx^2$, ſicque conſtat, quod iam ante innuimus, differentiale ſecundum ipſius y ad dx^2 habere rationem finitam.

128. In Capite primo iam notavimus differentias ſecundas atque ſequentes conſtitui non poſſe, niſi valores ſucceſſivi ipſius x certa quadam lege progredi aſſumantur, quae lex

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lex cum ſit arbitraria, his valoribus progreſſionem arithmeticam tanquam facillimam ſimulque aptiſſimam tribuimus. Ob eandem ergo rationem de differentialibus ſecundis nihil certi ſtatui poterit, niſi differentialia prima, quibus quantitas variabilis x continuo creſcere concipitur, ſecundum datam legem progrediantur; ponimus itaque differentialia prima ipſius x, nempe dx, dx^{I}, dx^{II}, &c. omnia inter ſe aequalia, unde fiunt differentialia ſecunda

$$ddx = dx^{\mathrm{I}} - dx = 0;\ ddx^{\mathrm{I}} = dx^{\mathrm{II}} - dx^{\mathrm{I}} = 0, \text{ \&c.}$$

Quoniam ergo differentialia ſecunda & ulteriora ab ordine, quem differentialia quantitatis variabilis x inter ſe tenent, pendent, hicque ordo ſit arbitrarius, quae conditio differentialia prima non afficit; hinc ingens diſcrimen inter differentialia prima ac ſequentia ratione inventionis intercedit.

129. Quodſi autem ſucceſſivi ipſius x valores x, x^{I}, x^{II}, x^{III}, x^{IV}, &c. non ſecundum arithmeticam progreſſionem ſtatuantur, ſed alia quacumque lege progredi ponantur, tum eorum quoque differentialia prima dx, dx^{I}, dx^{II}, &c. non erunt inter ſe aequalia, neque propterea erit $ddx = 0$. Hancobrem differentialia ſecunda quarumvis functionum ipſius x aliam formam induent; ſi enim huiuſmodi functionis y differentiale primum fuerit $= p\,dx$ ad eius differentiale ſecundum inveniendum non ſufficit differentiale ipſius p per dx multiplicaſſe, ſed inſuper ratio differentialis ipſius dx, quod eſt ddx haberi debet. Quoniam enim differentiale ſecundum oritur, ſi pdx a valore eius ſequente, qui oritur dum $x + dx$ loco x & $dx + ddx$ loco dx ponitur, ſubtrahatur ponamus valorem ipſius p ſequentem eſſe $= p + q\,dx$, eritque ipſius $p\,dx$ valor ſequens

$$= (p + qdx)\ (dx + ddx) = pdx + pddx + qdx^2 + qdxddx;$$

a quo ſubtrahatur pdx, eritque differentiale ſecundum

$$ddy = pddx + qdx^2 + qdxddx = pddx + qdx^2,$$

quia $qdxddx$ prae $pddx$ evaneſcit.

130.

130. Quanquam autem ratio aequalitatis eſt ſimpliciſſima atque aptiſſima, quae continuo ipſius x incrementis tribuatur, tamen frequenter evenire ſolet, ut non eius quantitatis variabilis x, cuius y eſt functio, incrementa aequalia aſſumantur, ſed alius cuiuſpiam quantitatis, cuius ipſa x ſit functio quaedam. Quin etiam ſaepe eiuſmodi alius quantitatis differentialia prima ſtatuuntur aequalia, cuius nequidem relatio ad x conſtet. Priori caſu pendebunt differentialia ſecunda & ſequentia ipſius x a ratione, quam x tenet ad illam quantitatem, quae aequabiliter creſcere ponitur, ex eaque pari modo definiri debent, quo hic differentialia ſecunda ipſius y ex differentialibus ipſius x definire docuimus. Poſteriori autem caſu differentialia ſecunda & ſequentia ipſius x tanquam incognita ſpectari, eorumque loco ſigna ddx, d^3x, d^4x, &c. uſurpari debebunt.

131. Cum autem, quemadmodum his caſibus differentiationes ſingulas abſolvi oporteat, infra fuſius ſimus oſtenſuri, hic pergamus quantitatem variabilem x tanquam uniformiter creſcentem aſſumere, ita ut eius differentialia prima dx, dx^{I}, dx^{II}, &c. inter ſe omnia aequalia, ac propterea differentialia ſecunda ac ſequentia nihilo aequalia ſtatuantur, quae conditio ita enunciari ſolet ut differentiale ipſius x nempe dx conſtans aſſumi dicatur. Sit deinde y functio quaecumque ipſius x, quae cum per x & conſtantes definiatur, ſingula quoque eius differentialia prima, ſecunda, tertia, quarta, &c. quae his ſignis indicantur dy, ddy, d^3y, d^4y, &c. per x & dx exprimi poterunt. Scilicet ſi in y loco x ſcribatur $x+dx$, ab hocque valore prior ſubtrahatur, remanebit differentiale primum dy: in quo ſi porro loco x ponatur $x+dx$, prodibit dy^{I}, eritque $ddy = dy^{I} - dy$, ſimili modo ponendo $x+dx$ loco x, ex ddy naſcetur ddy^{I}, atque $ddy^{I} - ddy$ dabit d^3y & ita porro: in quibus operationibus differentiale dx perpetuo tanquam quantitas conſtans ſpectatur, quae nullum differentiale recipiat.

132. Ex ratione, qua functio y per x determinatur, tam ope methodi differentiarum finitarum, quam multo ex-

peditius ex iis, quae poſtea ſumus tradituri, definietur valor functionis p, quae per dx multiplicata praebeat differentiale primum dy. Poſito ergo $dy = pdx$, differentiale ipſius pdx dabit differentiale ſecundum ddy; unde ſi fuerit $dp = qdx$, ob dx conſtans, orietur $ddy = qdx^2$, uti iam ante oſtendimus. Ulterius igitur progrediendo, cum differentialis ſecundi differentiale praebeat differentiale tertium, ponamus eſſe $dq = rdx$, eritque $d^3y = rdx^3$: ſimili modo ſi huius functionis r differentiale quaeratur, fueritque $dr = sdx$, habebitur differentiale quartum $d^4y = sdx^4$; ſicque porro, dummodo noverimus differentiale primum cuiuſque functionis invenire, differentiale cuiuſque ordinis aſſignare poterimus.

133. Quo igitur formae ſingulorum horum differentialium, ſimulque ratio ea inveniendi clarius menti repraeſentetur, ea ſequenti tabella complecti viſum eſt.

Si y fuerit functio quaecunque ipſius x,

atque poſito	erit
$dp = qdx$	$dy = pdx$
$dq = rdx$	$ddy = qdx^2$
$dr = sdx$	$d^3y = rdx^3$
$ds = tdx$	$d^4y = sdx^4$
&c.	$d^5y = tdx^5$

Cum igitur functio p ex functione y per differentiationem cognoſcatur, ſimilique modo ex p inveniatur q, hincque porro r, & ex eo ulterius s, &c. differentialia cuiuſvis ordinis ipſius y facile reperientur, dummodo differentiale dx aſſumatur conſtans.

134. Cum p, q, r, s, t, &c. ſint quantitates finitae, functiones nimirum ipſius x, differentiale primum ipſius y, rationem finitam habebit ad differentiale primum ipſius x, ſcilicet ut p ad 1; hancque ob cauſam differentialia dx & dy vocantur homogenea. Deinde cum ddy ad dx^2 habeat rationem finitam ut q ad 1, erunt ddy & dx^2 homogenea; ſimili modo homogenea erunt d^3y & dx^3, itemque d^4y & dx^4, & ita porro. Unde uti differentialia prima ſunt inter ſe homo-

mogenea, ſeu rationem finitam tenentia; ſic differentialia ſecunda cum quadratis differentialium primorum, differentialia autem tertia cum cubis differentialium primorum atque ita porro erunt homogenea. Atque generatim differentiale ipſius y ordinis n, quod ita exprimitur $d^n y$, homogeneum erit cum dx^n, hoc eſt cum poteſtate differentialis dx, cuius exponens eſt n.

135. Cum igitur prae dx evaneſcant omnes eius poteſtates, quarum exponentes ſunt unitate maiores, prae dy quoque evaneſcent dx^2, dx^3, dx^4, &c. & quae ad has poteſtates rationem finitam tenent differentialia altiorum ordinum ddy, d^3y, d^4y, &c. Simili modo prae ddy quia eſt homogeneum cum dx^2, omnes ipſius dx poteſtates quadrato ſuperiores dx^3, dx^4, &c. evaneſcent, evaneſcent ergo quoque d^3y, d^4y, &c. Atque prae d^3y, evaneſcent dx^4, d^4y; dx^5, d^5y, &c. Hincque facile, ſi propoſitae fuerint quaecunque expreſſiones huiuſmodi differentialia involventes, dignoſci poterunt, utrum ſint homogeneae nec ne. Reſpici enim debebunt tantum differentialia, omiſſis quantitatibus finitis, quippe quae homogeneitatem non turbant; atque pro differentialibus ſecundi altiorumque ordinum ſcribantur poteſtates ipſius dx ipſis homogeneae, quae ſi praebeant ubique eundem dimenſionum numerum, expreſſiones erunt homogeneae.

136. Ita patebit has expreſſiones $Pddy^2$ & $Qdyd^3y$ eſſe inter ſe homogeneas. Nam ddy^2 denotat quadratum ipſius ddy, & quia ddy homogeneum eſt cum dx^2, erit ddy^2 homogeneum cum dx^4. Deinde quia dy cum dx & d^3y cum dx^3 homogeneum eſt, erit productum dyd^3y cum dx^4 homogeneum: ex quo ſequitur expreſſiones $Pddy^2$ & $Qdyd^3y$ inter ſe eſſe homogeneas, ideoque rationem inter ſe finitam habere. Simili modo colligetur has expreſſiones $\frac{Pd^3y^2}{dxddy}$ & $\frac{Qd^5y}{dy^2}$ eſſe homogeneas; ſubſtitutis enim pro dy, ddy, d^3y & d^5y his ipſius dx poteſtatibus ipſis homogeneis dx, dx^2, dx^3, & dx^5, orientur hae expreſſiones Pdx^3 & Qdx^3, quae utique erunt inter ſe homogeneae.

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137. Quod ſi facta hac reductione expreſſiones propoſitae non contineant aequales ipſius dx poteſtates, tum non erunt homogeneae, neque propterea inter ſe rationem finitam tenebunt. Erit ergo altera infinities ſive maior ſive minor altera, hincque una reſpectu alterius evaneſcet. Sic $\frac{Pd^3y}{dx^2}$ ad $\frac{Qddy^2}{dy}$ rationem habebit infinite magnam: prior enim expreſſio reducitur ad Pdx & altera ad Qdx^3, unde haec prae illa evaneſcet. Quamobrem ſi in quopiam calculo aggregatum huiuſmodi binarum formularum occurrat, $\frac{Pd^3y}{dx^2} + \frac{Qddy^2}{dy}$, poſterior terminus prae priori tuto reiici, ſoluſque primus $\frac{Pd^3y}{dx^2}$ in calculo retineri poterit: ſubſiſtet enim perfecta ratio aequalitatis inter expreſſiones $\frac{Pd^3y}{dx^2} + \frac{Qddy^2}{dy}$ & $\frac{Pd^3y}{dx^2}$ quia exponens rationis eſt

$$= 1 + \frac{Qdx^2ddy^2}{Pdyd^3y} = 1 \quad \text{ob} \quad \frac{Qdx^2ddy^2}{Pdyd^3y} = 0.$$

Hocque pacto expreſſiones differentiales quandoque mirifice contrahi poſſunt.

138. In calculo differentiali praecepta traduntur, quorum ope cuiuſvis quantitatis propoſitae differentiale primum inveniri poteſt: & quoniam differentialia ſecunda ex differentiatione primorum, tertia per eandem operationem ex ſecundis & ita porro ſequentia ex praecedentibus reperiuntur, calculus differentialis continet methodum omnia cuiuſque ordinis differentialia inveniendi. Ex voce autem *differentialis*, qua differentia infinite parva denotatur, alia nomina derivantur, quae uſu ſunt recepta. Sic verbum habetur *differentiare*, quod ſignificat *differentiale invenire*, quantitaſque *differentiari* dicitur, quando eius differentiale elicitur. *Differentiatio* autem de-

no-

notat operationem, qua differentialia inveniuntur. Hinc calculus differentialis quoque vocatur methodus *differentiandi*, cum modum differentialia inveniendi contineat.

139. Quemadmodum in calculo differentiali cuiuſvis quantitatis differentiale inveſtigatur, ita viciſſim calculi ſpecies conſtituitur quoque in inventione eius quantitatis, cuius differentiale proponitur, qui calculus integralis vocatur. Si enim propoſitum fuerit differentiale quodcunque, eius reſpectu ea quantitas, cuius eſt differentiale, vocari ſolet integrale. Cuius denominationis ratio eſt, quod, cum differentiale conſiderari poſſit, tanquam pars infinite parva, qua quantitas quaepiam creſcit, ipſa illa quantitas reſpectu huius partis tanquam totum ſeu integrum ſpectari poteſt, hancque ob cauſam eius vocatur integrale. Sic cum dy ſit differentiale ipſius y, viciſſim y erit integrale ipſius dy, & cum ddy ſit differentiale ipſius dy, erit dy integrale ipſius ddy, ſimilique modo erit ddy integrale ipſius d^3y, & d^3y ipſius d^4y & ita porro: unde quaelibet differentiatio, ſi inverſe ſpectatur, integrationis exemplum exhibet.

140. Origo & natura integralium pariter ac differentialium clariſſime ex differentiarum finitarum doctrina in capite primo expoſita explicari poteſt. Poſtquam enim eſſet oſtenſum, quomodo cuiuſque quantitatis differentiam inveniri oporteat, retrogrediendo quoque monſtravimus, quomodo, ſi propoſita fuerit differentia, ea quantitas inveniri queat, cuius illa ſit differentia; quam quantitatem reſpectu ſuae differentiae vocavimus eius ſummam. Uti igitur ad infinite parva procedendo differentiae in differentialia abierunt, ita ſummae quae ibi erant vocatae, integralium nomen ſortiuntur: & hanc ob cauſam integralia quoque non raro ſummae appellari ſolent. Angli qui differentialia fluxiones nominant, integralia vocant quantitates fluentes; eorumque loquendi more datae fluxionis fluentem invenire, idem eſt, quod noſtro more dati differentialis integrale invenire dicimus.

141. Uti differentialia charactere d deſignamus, ita ad in-

integralia indicanda hac littera $\int$ utimur, quae ergo quantitatibus differentialibus praefixa eas denotabit quantitates, quarum illa ſunt differentialia. Sic ſi differentiale ipſius y fuerit pdx, ſeu $dy = pdx$, erit y integrale ipſius pdx, quod hoc modo ſcribitur $y = \int pdx$, cum ſit $y = \int dy$. Integrale ergo ipſius pdx, quod per $\int pdx$ indicatur, denotat quantitatem, cuius differentiale eſt pdx. Simili modo cum ſit $ddy = qdx^2$ exiſtente $dp = qdx$; erit integrale ipſius ddy hoc eſt $dy = pdx$, atque ob $p = \int qdx$, erit $dy = dx\int qdx$, ac propterea $y = \int dx\int qdx$. Si ulterius ſit $dq = rdx$, erit $q = \int rdx$ & $dp = dx\int rdx$; unde ſi character $\int$ denuo praefigatur, fiet $p = \int dx\int rdx$, porroque $dy = dx\int dx\int rdx$, atque $y = \int dx\int dx\int rdx$.

142. Quia differentiale dy eſt quantitas infinite parva, eius integrale autem y quantitas finita, parique modo differentiale ſecundum ddy infinities minus eſt, quam eius integrale dy, manifeſtum eſt differentialia prae ſuis integralibus evaneſcere. Quae affectio quo melius percipiatur, infinite parva in ordines dividi ſolent, diciturque infinite parvum primi ordinis, ad quod referuntur differentialia prima dx, dy. Infinite parvum ſecundi ordinis complectitur differentialia ſecundi ordinis, quae homogenea ſunt cum dx^2; ſimilique modo infinite parva, quae cum dx^3 ſunt homogenea, vocantur ordinis tertii, ad quem ergo pertinent differentialia tertia omnia; ſicque porro. Unde uti infinite parva primi ordinis prae quantitatibus finitis evaneſcunt, ſic infinite parva ſecundi ordinis prae infinite parvis primi ordinis, atque generatim infinite parva cuiuſque ordinis altioris prae infinite parvis ordinis inferioris evaneſcent.

143. His igitur infinite parvorum ordinibus conſtitutis, uti differentiale quantitatis finitae eſt infinite parvum primi ordinis, atque differentiale infinite parvi primi ordinis eſt infinite parvum ſecundi ordinis, & ita porro; ita viciſſim manifeſtum eſt integrale infinite parvi primi ordinis eſſe quantitatem finitam, integrale autem infinite parvi ſecundi ordinis eſſe infinite parvum primi ordinis ſicque deinceps.

Qua-

Quare ſi differentiale propoſitum fuerit infinite parvum ordinis n, eius integrale erit infinite parvum ordinis $n-1$; hincque uti differentiando ordo infinite parvorum augetur, ita integratione ad ordines inferiores progredimur, donec ad ipſas quantitates finitas perveniamus. Sin autem quantitates finitas denuo integrare velimus, tum ſecundum hanc legem perveniemus ad quantitates infinite magnas, ab harumque integratione inſtituta ad quantitates adhuc infinities maiores, ſicque progrediendo obtinebimus ſimiles infinitorum ordines, quorum quisque praecedentem infinities ſuperat.

144. Supereſt ut in hoc Capite quaedam de uſu ſignorum recepto moneamus, ne ambiguitati ullus locus relinquatur. Ac primo quidem ſignum differentiationis d tantum afficit litteram immediate ſequentem ſolam: ſic dxy non denotat differentiale producti xy, ſed differentiale ipſius x per ipſam quantitatem y multiplicatum. Solet autem, quominus confuſio naſcatur, quantitas y ante ſignum d hoc modo ſcribi ydx, quo productum ex y in dx indicatur. Attamen ſi y ſit quantitas vel ſignum radicale $\surd$ vel logarithmicum habens praefixum, tum poſt differentiale poni ſolet: nimirum $dx\surd(aa-xx)$ ſignificat productum ex quantitate finita $\surd(aa-xx)$ in differentiale dx, ſimilique modo $dxl(1+x)$ eſt productum ex logarithmo quantitatis $1+x$, per dx multiplicato. Ob eandem rationem $ddy\surd x$ exprimit productum differentialis ſecundi ddy & quantitatis finitae $\surd x$.

145. Neque vero ſignum d litteram immediate ſequentem ſolam afficit, ſed etiam nequidem exponentem, ſi quem habet, ſpectat. Ita dx^2 non exprimit differentiale ipſius x^2, ſed quadratum differentialis ipſius x, ita ut exponens 2 non ad x, ſed ad dx referri debeat. Poſſet etiam ſcribi $dxdx$, quemadmodum productum duorum differentialium dx & dy hoc modo $dxdy$ exponitur, verum prior modus dx^2, uti eſt brevior, ita uſitatior. Praeſertim ſi altiores poteſtates ipſius dx eſſent indicandae, nimis prolixum foret dx toties repeti: ſic dx^3 denotat cubum ipſius dx, & in differentialibus altio-

rum

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rum ordinum ſimilis ratio obſervatur. Scilicet ddy^4 denotat poteſtatem quartam differentialis ſecundi ordinis ddy; atque $d^3y^2\sqrt{x}$ ſignificat quadratum differentialis tertii ordinis ipſius y multiplicatum eſſe per $\sqrt{x}$; ſin autem per quantitatem rationalem x multiplicari deberet, ea praefigitur hoc modo xd^3y^2.

146. Sin autem velimus, ut ſignum d plus quam ſolam litteram ſubſequentem afficiat, id peculiari modo indicari debet. Utimur hoc caſu praecipue uncinulis, quibus ea quantitas includitur, cuius differentiale debet indicare. Uti $d(xx+yy)$ denotat differentiale quantitatis $xx+yy$; verum ſi velimus differentiale poteſtatis huiuſmodi quantitatis deſignare, ambiguitatem vix evitare poſſumus: ſi enim ſcribamus $d(xx+yy)^2$, intelligi poſſet quadratum ipſius $d(xx+yy)$. Poterimus autem hoc caſu punctum in auxilium vocare, ita ut $d.(xx+yy)^2$ denotet differentiale ipſius $(xx+yy)^2$, omiſſo autem puncto $d(xx+yy)^2$ quadratum ipſius $d(xx+yy)$. Puncto ſcilicet commode indicari poteſt ſignum d ad totam quantitatem poſt punctum ſequentem pertinere: ſic $d.xdy$ exprimet differentiale ipſius xdy; & $d.^3xdy\sqrt{(aa+xx)}$ differentiale tertii ordinis expreſſionis $xdy\sqrt{(aa+xx)}$, quae eſt productum ex quantitatibus finitis x & $\sqrt{(aa+xx)}$ atque ex differentiali dy.

147. Quemadmodum autem ſignum differentiationis d ſolam quantitatem immediate ſequentem afficit, niſi puncto interpoſito eius vis ad totam expreſſionem ſequentem extendatur; ita contra ſignum integrationis $\int$ ſemper totam expreſſionem, cui eſt praefixum, complectitur. Ita $\int ydx(aa-xx)^n$ denotat integrale ſeu eam quantitatem, cuius differentiale eſt $ydx(aa-xx)^n$, atque haec expreſſio $\int xdx\int dxlx$ denotat quantitatem, cuius differentiale eſt $xdx\int dxlx$. Hinc ſi velimus productum duorum integralium ſcilicet $\int ydx$ & $\int zdx$ exprimere, id hoc modo $\int ydx\int zdx$ perperam fiet, intelligeretur enim integrale quantitatis $ydx\int zdx$. Hanc ob cauſam iterum puncto ſolet haec ambiguitas tolli, ita ut $\int ydx\,.\int zdx$ ſignificet productum integralium $\int ydx$ & $\int zdx$.

148. Analyſis infinitorum igitur cum in differentialibus

bus tum in integralibus inveniendis verſatur, & hancobrem in duas praecipuas partes dividitur, quarum altera vocatur Calculus differentialis, altera Calculus integralis. In priori praecepta traduntur, quantitatum quarumvis differentialia inveniendi; in poſteriori vero via monſtratur differentialium propoſitorum integralia inveſtigandi: in utroque autem ſimul ſummus uſus, quem iſti calculi tam ad ipſam Analyſin quam ad Geometriam ſublimiorem afferunt, indicatur. Quam ob cauſam iſta Analyſeos pars iam tanta accepit incrementa, ut modico volumine prorſus comprehendi nequeat. Imprimis vero in calculo integrali indies tam nova artificia integrandi, quam adiumenta eius in ſolvendis varii generis problematibus, deteguntur, ut ob haec nova inventa, quae continuo accedunt, nunquam exhauriri, multo minus perfecte deſcribi atque explicari poſſit. Dabo autem operam, ut quae adhuc ſunt reperta, vel cuncta in his libris exponam, vel ſaltem methodos explicem, unde ea facile deduci queant.

149. Solent vulgo plures Analyſeos infinitorum partes numerari; praeter calculos enim differentialem & integralem inveniuntur paſſim calculi differentio-differentialis atque exponentialis. In calculo differentio-differentiali tradi ſolet methodus differentialia ſecundi atque altiorum ordinum inveniendi: quoniam autem modum cuiuſque ordinis differentialia inveniendi in ipſo calculo differentiali ſum expoſiturus, hac ſubdiviſione, quae potius ex merito inventionis, quam ex re ipſa facta eſſe videtur, ſuperſedebimus. Quod deinde ad calculum exponentialem attinet, quo Celeb. IOH. BERNOULLI, cui ob innumera eaque maxima incrementa Analyſeos infinitorum aeternas debemus gratias, methodos differentiandi atque integrandi ad quantitates exponentiales tranſtulit, quia utrumque calculum ad omnis generis quantitates tam algebraicas quam tranſcendentes accommodare conſtitui, hinc partem peculiarem facere ſuperfluum atque inſtituto contrarium foret.

N 150.

150. Primum igitur calculum differentialem in hoc libro pertractare ſtatui, modumque ſum expoſiturus, cuius ope omnium quantitatum variabilium differentialia non ſolum prima, ſed etiam ſecunda & altiorum ordinum expedite inveniri queant. Primum ergo quantitates algebraicas contemplabor, ſive ſint functiones unius variabilis, ſive plurium, ſive demum explicite dentur, ſive per aequationes. Deinde inventionem differentialium quoque accommodabo ad quantitates non algebraicas, ad quarum notitiam quidem ſine calculi integralis ſubſidio pervenire licet: cuiuſmodi ſunt logarithmi, atque quantitates exponentiales; deinde etiam arcus circuli, viciſſimque arcuum circularium ſinus, & tangentes. Denique etiam quantitates utcunque ex his compoſitas & permixtas differentiare docebo; ſicque calculi differentialis pars prior, methodus ſcilicet differentiandi abſolvetur.

151. Altera pars uſui, quem methodus differentiandi tam ad Analyſin quam Geometriam ſublimiorem affert, explicando eſt deſtinata. In Algebram autem communem inde plurima redundant commoda, partim ad radices aequationum inveniendas, partim ad ſeries tractandas atque ſummandas, partim ad maxima minimaque eruenda, partim ad valores expreſſionum, quae certis caſibus indeterminatae videantur, definiendos, & quae ſunt alia. Geometria autem ſublimior ex calculo differentiali maxima accepit incrementa, dum eius ope tangentes linearum curvarum, earumque curvatura ipſa mira facilitate definiri, multaque alia problemata circa radios a lineis curvis vel reflexos vel refractos reſolvi poſſunt. Quibus etſi ampliſſimus tractatus impleri poſſet, tamen conabor, quantum fieri licet, omnia breviter ac perſpicue explicare.

CA-

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CAPUT V.

DE DIFFERENTIATIONE FUNCTIONUM ALGEBRAICARUM UNICAM VARIABILEM INVOLVENTIUM.

152.

Quia quantitatis variabilis x differentiale eſt $= dx$ erit x in proximum promovendo $x^I = x + dx$. Quare ſi fuerit y quaecunque functio ipſius x, ſi in ea loco x ponatur $x + dx$, ea abibit in y^I, atque differentia $y^I - y$ dabit differentiale ipſius y. Si igitur ponamus $y = x^n$ fiet

$$y^I = (x + dx)^n = x^n + nx^{n-1}dx + \frac{n(n-1)}{1.\ 2} x^{n-2} dx^2 + \text{\&c.}$$

eritque ergo

$$dy = y^I - y = nx^{n-1}dx + \frac{n(n-1)}{1.\ 2} x^{n-2} dx^2 + \text{\&c.}$$

At in hac expreſſione terminus ſecundus cum reliquis ſequentibus prae primo evaneſcit, eritque idcirco $nx^{n-1}dx$ differentiale ipſius x^n, ſeu $d.x^n = nx^{n-1}dx$. Unde ſi a ſit numerus ſeu quantitas conſtans, erit quoque $d.ax^n = nax^{n-1}dx$. Cuiuſcunque ergo ipſius x poteſtatis differentiale invenitur, multiplicando eam per exponentem, dividendo per x, & reliquum per dx multiplicando, quae regula facile memoria retinetur.

153. Cognito differentiali primo ipſius x^n, ex eo facile differentiale ſecundum reperitur, dummodo, ut hic conſtanter aſſumemus, differentiale dx conſtans ſtatuatur. Cum enim in differentiali $nx^{n-1}dx$ factor ndx ſit conſtans, alterius factoris x^{n-1} differentiale ſumi debet, quod proinde erit $(n-1)x^{n-2}dx$. Hoc ergo per ndx multiplicatum dabit diffe-

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rentiale ſecundum : $dd.x^n = n(n-1)x^{n-2}dx^2$. Simili modo ſi differentiale ipſius x^{n-2} quod eſt $= (n-2)x^{n-3}dx$ multiplicetur per $n(n-1)dx^2$ prodibit differentiale tertium

$$d.^3 x^n = n(n-1)\ (n-2)x^{n-3}dx^3.$$

Porro itaque erit differentiale quartum

$$d.^4 x^n = n(n-1)\ (n-2)\ (n-3)x^{n-4}dx^4,$$

& differentiale quintum

$$d.^5 x^n = n(n-1)\ (n-2)\ (n-3)\ (n-4)x^{n-5}dx^5;$$

unde ſimul forma ſequentium differentialium facillime colligitur.

154. Quoties ergo n eſt numerus integer affirmativus, toties ad differentialia tandem pervenitur evaneſcentia; quae ſcilicet ita ſunt $=0$, ut prae omnibus ipſius dx poteſtatibus evaneſcant. Horum autem notandi ſunt caſus ſimpliciores.

$d.x = dx$; $dd.x = 0$; $d.^3 x = 0$; &c.

$d.x^2 = 2xdx$; $dd.x^2 = 2dx^2$; $d.^3 x^2 = 0$; $d.^4 x^2 = 0$ &c.

$d.x^3 = 3x^2 dx$; $dd.x^3 = 6xdx^2$; $d.^3 x^3 = 6dx^3$; $d.^4 x^3 = 0$

$d.x^4 = 4x^3 dx$; $dd.x^4 = 12x^2 dx^2$; $d.^3 x^4 = 24xdx^3$; $d.^4 x^4 = 24dx^4$

$d.x^5 = 5x^4 dx$; $dd.x^5 = 20x^3 dx^2$; $d.^3 x^5 = 60x^2 dx^3$; $d.^4 x^5 = 120xdx^4$; $d.^5 x^5 = 120dx^5$; $d.^6 x^5 = 0$.

Patet ergo ſi n fuerit numerus integer affirmativus, poteſtatis x^n differentiale ordinis n eſſe conſtans, nempe $= 1.2.3.\ldots n dx^n$, adeoque differentialia ſuperiorum ordinum omnium eſſe $=0$.

155. Si n ſit numerus integer negativus, huiuſmodi ipſius x poteſtatum negativarum $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$, &c. differentialia ſumi poterunt, cum ſit $\frac{1}{x} = x^{-1}$; $\frac{1}{xx} = x^{-2}$, & generaliter $\frac{1}{x^m} = x^{-m}$. Si ergo in formula antecedente ponatur

$n = -m$, erit ipſius $\frac{1}{x^m}$ differentiale primum $= \frac{-m\,dx}{x^{m+1}}$; differentiale ſecundum $= \frac{m(m+1)dx^2}{x^{m+2}}$; differentiale tertium $= \frac{-m(m+1)(m+2)dx^3}{x^{m+3}}$ &c. unde ſequentes caſus ſimpliciores imprimis notari merentur.

$$d.\frac{1}{x} = \frac{-dx}{x^2};\quad dd.\frac{1}{x} = \frac{2dx^2}{x^3};\quad d.^3\frac{1}{x} = \frac{-6dx^3}{x^4}$$
$$d.\frac{1}{x^2} = \frac{-2dx}{x^3};\quad dd.\frac{1}{x^2} = \frac{6dx^2}{x^4};\quad d.^3\frac{1}{x^2} = \frac{-24dx^3}{x^5}$$
$$d.\frac{1}{x^3} = \frac{-3dx}{x^4};\quad dd.\frac{1}{x^3} = \frac{12dx^2}{x^5};\quad d.^3\frac{1}{x^3} = \frac{-60dx^3}{x^6}$$
$$d.\frac{1}{x^4} = \frac{-4dx}{x^5};\quad dd.\frac{1}{x^4} = \frac{20dx^2}{x^6};\quad d.^3\frac{1}{x^4} = \frac{-120dx^3}{x^7}$$
$$d.\frac{1}{x^5} = \frac{-5dx}{x^6};\quad dd.\frac{1}{x^5} = \frac{30dx^2}{x^7};\quad d.^3\frac{1}{x^5} = \frac{-210dx^3}{x^8}$$

&c.

156. Ponendis deinde pro n numeris fractis differentialia formularum irrationalium obtinebimus. Sit enim

$n = \frac{\mu}{\nu}$, erit formulae $x^{\frac{\mu}{\nu}}$ ſeu $\sqrt[\nu]{x^\mu}$ differentiale primum

$$= \frac{\mu}{\nu} x^{\frac{\mu-\nu}{\nu}} dx = \frac{\mu}{\nu} dx \sqrt[\nu]{x^{\mu-\nu}}$$ ſecundum

$$= \frac{\mu(\mu-\nu)}{\nu^2} x^{\frac{\mu-2\nu}{\nu}} dx^2 = \frac{\mu(\mu-\nu)}{\nu\nu} dx^2 \sqrt[\nu]{x^{\mu-2\nu}}$$ &c.

Hinc

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Hinc erit:

$$d.\sqrt{x} = \frac{dx}{2\sqrt{x}}\;;\quad dd.\sqrt{x} = \frac{-dx^2}{4x\sqrt{x}}\;;\quad d^3\sqrt{x} = \frac{1.3dx^3}{8x^2\sqrt{x}}$$

$$d.\sqrt[3]{x} = \frac{dx}{3\sqrt[3]{x^2}}\;;\quad dd.\sqrt[3]{x} = \frac{-2dx^2}{9x\sqrt[3]{x^2}}\;;\quad d^3\sqrt[3]{x} = \frac{2.5dx^3}{27x^2\sqrt[3]{x^2}}$$

$$d.\sqrt[4]{x} = \frac{dx}{4\sqrt[4]{x^3}}\;;\quad dd.\sqrt[4]{x} = \frac{-3dx^2}{16x\sqrt[4]{x^3}}\;;\quad d^3\sqrt[4]{x} = \frac{3.7dx^3}{64x^2\sqrt[4]{x^3}}$$

quae expreſſiones ſi pauliſper inſpiciantur, facile habitus acquiretur huiuſmodi differentialia, etiam ſine praevia reductione ad formam poteſtatis, inveniendi.

157. Si μ non fuerit 1, ſed numerus alius ſive affirmativus ſive negativus integer, differentialia aeque facile definientur. Cum autem differentialia ſecunda & altiorum ordinum eadem lege ex primis, qua haec ex ipſis poteſtatibus, deriventur, exempla ſimpliciora primorum tantum differentialium apponamus.

$$d.x\sqrt{x} = \tfrac{3}{2}dx\sqrt{x};\; d.x^2\sqrt{x} = \tfrac{5}{2}xdx\sqrt{x};\; d.x^3\sqrt{x} = \tfrac{7}{2}x^2dx\sqrt{x};$$

$$d.\frac{1}{\sqrt{x}} = \frac{-dx}{2x\sqrt{x}};\quad d.\frac{1}{x\sqrt{x}} = \frac{-3dx}{2xx\sqrt{x}};\quad d.\frac{1}{xx\sqrt{x}} = \frac{-5dx}{2x^3\sqrt{x}};$$

$$d.\sqrt[3]{x^2} = \tfrac{2}{3}\frac{dx}{\sqrt[3]{x}};\quad d.x\sqrt[3]{x} = \tfrac{4}{3}dx\sqrt[3]{x};\quad d.x\sqrt[3]{x^2} = \tfrac{5}{3}dx\sqrt[3]{x^2};$$

$$d.xx\sqrt[3]{x} = \tfrac{7}{3}xdx\sqrt[3]{x};\quad d.xx\sqrt[3]{x^2} = \tfrac{8}{3}xdx\sqrt[3]{x^2};\quad \&c.$$

$$d.\frac{1}{\sqrt[3]{x}} = \frac{-dx}{3x\sqrt[3]{x}};\quad d.\frac{1}{\sqrt[3]{x^2}} = \frac{-2dx}{3x\sqrt[3]{x^2}};\quad d.\frac{1}{x\sqrt[3]{x}} = \frac{-4dx}{3x^2\sqrt[3]{x}};$$

$$d.\frac{1}{x\sqrt[3]{x^2}} = \frac{-5dx}{3x^2\sqrt[3]{x^2}};\quad d.\frac{1}{x^2\sqrt[3]{x}} = \frac{-7dx}{3x^3\sqrt[3]{x}};\quad \&c.$$

158. Ex his iam functionum omnium algebraicarum rationalium integrarum differentialia poterunt inveniri, propterea quod earum ſinguli termini ſunt poteſtates ipſius x,

quas

quas differentiare novimus. Cum enim quantitas huiuſmodi $p+q+r+s+$ &c. poſito $x+dx$ loco x abeat in $p+dp+q+dq+r+dr+s+ds+$ &c. erit eius differentiale $=dp+dq+dr+ds+$ &c. Quare ſi ſingularum quantitatum p, q, r, s, differentialia aſſignare queamus, ſimul quoque aggregati earum differentiale innoteſcet. Atque cum multipli ipſius p differentiale ſit aeque multiplum ipſius dp, hoc eſt $d.\,ap=adp$; erit quantitatis $ap+bq+cr$ differentiale $=adp+bdq+cdr$. Cum denique quantitatum conſtantium differentialia ſint nulla, erit quoque quantitatis huius $ap+bq+cr+f$ differentiale $=adp+bdq+cdr$.

159. In functionibus ergo rationalibus integris cum ſinguli termini ſint vel conſtantes vel poteſtates ipſius x, differentiatio ſecundum praecepta data facile abſolvetur. Sic erit:

$$d(a+x)=dx\quad;\quad d(a+bx)=bdx;$$
$$d(a+xx)=2xdx\quad;\quad d(aa-xx)=-2xdx;$$
$$d(a+bx+cxx)=bdx+2cxdx;$$
$$d(a+bx+cxx+ex^3)=bdx+2cxdx+3ex^2dx;$$
$$d(a+bx+cxx+ex^3+fx^4)=bdx+2cxdx+3ex^2dx+4fx^3dx.$$

Atque ſi exponentes fuerint indefiniti erit:

$$d(1-x^n)=-nx^{n-1}dx\quad;\quad d(1+x^m)=mx^{m-1}dx;$$
$$d(a+bx^m+cx^n)=mbx^{m-1}dx+ncx^{n-1}dx.$$

160. Cum igitur functiones rationales integrae ſecundum maximam ipſius x dignitatem in gradus diſtinguantur, manifeſtum eſt, ſi huiuſmodi functionum continuo differentialia capiantur, ea tandem fieri conſtantia, poſteaque in nihilum abire, ſi quidem differentiale dx aſſumatur conſtans. Sic functionis primi gradus $a+bx$ differentiale primum bdx eſt conſtans, ſecundum cum ſequentibus nullum. Sit functio ſecundi gradus

$$a+bx+cxx=y;\ \text{erit}\ dy=bdx+2cxdx;$$
$$ddy=2cdx^2;\ d^3y=0.$$

Si-

Simili modo ſi ponatur functio tertii gradus $a+bx+cxx+ex^3=y$; erit $dy=bdx+2cxdx+3exxdx$; $ddy=2cdx^2+6exdx^2$ & $d^3y=6edx^3$ atque $d^4y=0$. Quare generaliter ſi huiuſmodi functio ſit gradus n, eius differentiale ordinis n erit conſtans, & ſequentia omnia nulla.

161. Neque etiam differentiatio turbabitur, ſi inter poteſtates ipſius x, quae huiuſmodi functionem componunt, occurrant tales, quarum exponentes ſint numeri negativi ſeu fracti. Ita

I. Si ſit $y=a+b\sqrt{x}-\frac{c}{x}$

erit $dy=\frac{bdx}{2\sqrt{x}}+\frac{cdx}{xx}$.

II. Si ſit $y=\frac{a}{\sqrt{x}}+b+c\sqrt{x}-ex$

erit $dy=\frac{-adx}{2x\sqrt{x}}+\frac{cdx}{2\sqrt{x}}-edx$,

& $ddy=\frac{3adx^2}{4xx\sqrt{x}}-\frac{cdx^2}{4x\sqrt{x}}$.

III. Si ſit $y=a+\frac{b}{\sqrt[3]{xx}}-\frac{c}{x\sqrt[3]{x}}+\frac{f}{xx}$

erit $dy=\frac{-2bdx}{3x\sqrt[3]{xx}}+\frac{4cdx}{3xx\sqrt[3]{x}}-\frac{2fdx}{x^3}$

& $ddy=\frac{10bdx^2}{9x^2\sqrt[3]{xx}}-\frac{28cdx^2}{9x^3\sqrt[3]{x}}+\frac{6fdx^2}{x^4}$.

cuiuſmodi exempla ſecundum praecepta data facillime abſolvuntur.

161.

162. Si quantitas differentianda propoſita fuerit poteſtas eiuſmodi functionis, cuius differentiale exhibere valemus, praecedentia praecepta ſufficiunt ad eius differentiale primum definiendum. Sit enim p functio quaecunque ipſius x, cuius differentiale dp in poteſtate eſt, erit ipſius poteſtatis p^n differentiale primum $= np^{n-1}dp$. Hinc ſequentia exempla ſolvuntur:

I. Si ſit $y = (a+x)^n$; erit $dy = n(a+x)^{n-1}dx$

II. Si ſit $y = (aa-xx)^2$; erit $dy = -4xdx(aa-xx)$

III. Si ſit $y = \frac{1}{aa+xx}$ ſeu $y = (aa+xx)^{-1}$

erit $dy = \frac{-2xdx}{(aa+xx)^2}$.

IV. Si ſit $y = \sqrt{(a+bx+cxx)}$; erit $dy = \frac{bdx+2cxdx}{2\sqrt{(a+bx+cxx)}}$.

V. Si ſit $y = \sqrt[3]{(a^4-x^4)^2}$ ſeu $y = (a^4-x^4)^{\frac{2}{3}}$

erit $dy = -\frac{8}{3}x^3dx(a^4-x^4)^{-\frac{1}{3}} = \frac{-8x^3dx}{3\sqrt[3]{(a^4-x^4)}}$.

VI. Si ſit $y = \frac{1}{\sqrt{(1-xx)}}$ ſeu $y = (1-xx)^{-\frac{1}{2}}$

erit $dy = xdx(1-xx)^{-\frac{3}{2}} = \frac{xdx}{(1-xx)\sqrt{(1-xx)}}$;

VII. Si ſit $y = \sqrt[3]{(a+\sqrt{bx}+x)}$

erit $dy = \frac{dx\sqrt{b}:2\sqrt{x}+dx}{3\sqrt[3]{(a+\sqrt{bx}+x)^2}} = \frac{dx\sqrt{b}+2dx\sqrt{x}}{6\sqrt{x}\cdot\sqrt[3]{(a+\sqrt{bx}+x)^2}}$;

VIII. Si ſit $y = \frac{1}{x+\sqrt{(aa-xx)}}$,

ob $d.\sqrt{(aa-xx)} = \frac{-xdx}{\sqrt{(aa-xx)}}$, erit

$$dy = \frac{-dx+xdx:\sqrt{(aa-xx)}}{(x+\sqrt{(aa-xx)})^2} = \frac{xdx-dx\sqrt{(aa-xx)}}{(x+\sqrt{(aa-xx)})^2\sqrt{(aa-xx)}}$$

ſeu $dy = \frac{dx(x-\sqrt{(aa-xx)})^3}{(2xx-aa)^2\sqrt{(aa-xx)}}$.

IX. Si ſit $y = \sqrt[4]{\left(1-\frac{1}{\sqrt{x}}+\sqrt[3]{(1-xx)^2}\right)^3}$.

Ponatur $\frac{1}{\sqrt{x}} = p$ & $\sqrt[3]{(1-xx)^2} = q$;

ob $y = \sqrt[4]{(1-p+q)^3}$, erit $dy = \frac{-3dp+3dq}{4\sqrt[4]{(1-p+q)}}$.

Iam per antecedentia eſt $dp = \frac{-dx}{2x\sqrt{x}}$ & $dq = \frac{-4xdx}{3\sqrt[3]{(1-xx)}}$,

quibus valoribus ſubſtitutis fiet:

$$dy = \frac{3dx:2x\sqrt{x}-4xdx:\sqrt[3]{(1-xx)}}{4\sqrt[4]{\left(1-\frac{1}{\sqrt{x}}+\sqrt[3]{(1-xx)^2}\right)}}.$$

Simili autem modo ſingulares litteras loco terminorum aliquantum compoſitorum ſubſtituendo omnium huiuſmodi functionum differentialia facile eruuntur.

163. Si quantitas differentianda fuerit productum ex duabus pluribuſve functionibus ipſius x, quarum differentialia conſtant, eius differentiale ſequente modo commodiſſime invenietur. Sint p & q functiones ipſius x, quarum differentialia dp & dq iam ſunt cognita, quia poſito $x+dx$ loco x; p abit in $p+dp$ & q in $q+dq$: productum pq tranſmutabitur in $(p+dp)(q+dq) = pq+pdq+qdp+dpdq$. Unde pro-

producti pq differentiale erit $= pdq + qdp + dpdq$; ubi cum pdq & qdp ſint infinite parva primi ordinis, at $dpdq$ ſecundi ordinis, ultimus terminus evaneſcet, eritque igitur $d.pq = pdq + qdp$. Differentiale ergo producti pq conſtat ex duobus membris, quae obtinentur, ſi uterque factor per differentiale alterius factoris multiplicetur. Hinc facile deducitur differentiatio producti pqr ex tribus factoribus conſtantis: ponatur enim $qr = z$, fiet $pqr = pz$, & $d.pqr = pdz + zdp$, verum ob $z = qr$ erit $dz = qdr + rdq$, quibus valoribus loco z & dz ſubſtitutis erit $d.pqr = pqdr + prdq + qrdp$.
Simili modo ſi quantitas differentianda quatuor habeat factores erit: $d.pqrs = pqrds + pqsdr + prsdq + qrsdp$:
unde quilibet differentiationem plurium factorum facile perſpiciet.

I. Si ergo fuerit $y = (a+x)(b-x)$, erit

$$dy = -dx(a+x) + dx(b-x) = -adx + bdx - 2xdx$$

quod idem differentiale quoque invenitur, ſi quantitas propoſita evolvatur: fit enim $y = ab - ax + bx - xx$, ideoque per ſuperiora praecepta $dy = -adx + bdx - 2xdx$.

II. Si fuerit $y = \frac{1}{x}\surd(aa-xx)$.

Ponatur $\frac{1}{x} = p$ & $\surd(aa-xx) = q$, quia eſt $dp = \frac{-dx}{xx}$

$$\& \ dq = \frac{-xdx}{\surd(aa-xx)}, \text{ erit}$$

$$dy = pdq + qdp = \frac{-dx}{\surd(aa-xx)} - \frac{dx}{xx}\surd(aa-xx);$$

quae ad eundem denominatorem reductae dabunt

$\frac{-xxdx - aadx + xxdx}{xx\surd(aa-xx)} = \frac{-aadx}{xx\surd(aa-xx)}$. Hinc erit differentiale quaeſitum, $dy = \frac{-aadx}{xx\surd(aa-xx)}$.

 III.

III. Si fuerit $y = \frac{xx}{\sqrt{(a^4+x^4)}}$.

Ponatur $xx = p$, & $\frac{1}{\sqrt{(a^4+x^4)}} = q$; quia invenimus $dp = 2xdx$ & $dq = \frac{-2x^3dx}{(a^4+x^4)^{\frac{3}{2}}}$, erit

$$pdq+qdp = \frac{-2x^5dx}{(a^4+x^4)^{\frac{3}{2}}} + \frac{2xdx}{\sqrt{(a^4+x^4)}} = \frac{2a^4xdx}{(a^4+x^4)^{\frac{3}{2}}},$$

Hinc ergo erit differentiale quaeſitum

$$dy = \frac{2a^4xdx}{(a^4+x^4)\sqrt{(a^4+x^4)}}.$$

IV. Si fuerit $y = \frac{x}{x+\sqrt{(1+xx)}}$.

Ponendo $x = p$ & $\frac{1}{x+\sqrt{(1+xx)}} = q$, ob $dp = dx$ & $dq = \frac{-dx-xdx:\sqrt{(1+xx)}}{(x+\sqrt{(1+xx)})^2} = \frac{-dx(x+\sqrt{(1+xx)})}{(x+\sqrt{(1+xx)})^2\sqrt{(1+xx)}}$ $= \frac{-dx}{(x+\sqrt{(1+xx)})\sqrt{(1+xx)}}$, erit $pdq+qdp =$ $= \frac{-xdx}{(x+\sqrt{(1+xx)})\sqrt{(1+xx)}} + \frac{dx}{x+\sqrt{(1+xx)}} =$ $= \frac{dx(\sqrt{(1+xx)}-x)}{(x+\sqrt{(1+xx)})\sqrt{(1+xx)}}$. Fiet ergo differentiale quaeſitum $dy = \frac{dx(\sqrt{(1+xx)}-x)}{(x+\sqrt{(1+xx)})\sqrt{(1+xx)}}$; cuius fractionis ſi numerator ac denominator multiplicetur per $\sqrt{(1+xx)}-x$, fiet $dy = \frac{dx(1+2xx-2x\sqrt{(1+xx)})}{\sqrt{(1+xx)}} =$

dx

$$\frac{dx+2xxdx}{\sqrt{(1+xx)}}-2xdx.$$

Idem differentiale alio modo commodius inveniri poteſt; cum enim ſit $y=\frac{x}{x+\sqrt{(1+xx)}}$, multiplicetur numerator ac denominator per $\sqrt{(1+xx)}-x$, fietque

$$y=x\sqrt{(1+xx)}-xx=\sqrt{(x^2+x^4)}-xx,$$

cuius differentiale per priorem regulam eſt

$$dy=\frac{xdx+2x^3dx}{\sqrt{(xx+x^4)}}-2xdx=\frac{dx+2xxdx}{\sqrt{(1+xx)}}-2xdx.$$

V. Si fuerit $y=(a+x)(b-x)(x-c)$, erit

$$dy=(a+x)(b-x)dx-(a+x)(x-c)dx+(b-x)(x-c)dx.$$

VI. Si fuerit $y=x(aa+xx)\sqrt{(aa-xx)}$.

Ob tres factores ergo reperietur

$$dy=dx(aa+xx)\sqrt{(aa-xx)}+2xxdx\sqrt{(aa-xx)}$$
$$-\frac{xxdx(aa+xx)}{\sqrt{(aa-xx)}}=\frac{dx(a^4+aaxx-4x^4)}{\sqrt{(aa-xx)}}.$$

164. Quanquam etiam fractiones in factoribus comprehendi poſſunt, tamen commodius utemur regula fractionibus differentiandi inſerviente. Sit ergo propoſita haec fractio $\frac{p}{q}$; cuius differentiale inveniri oporteat. Quoniam poſito $x+dx$ loco x fractio illa abit in

$$\frac{p+dp}{q+dq}=(p+dp)\left(\frac{1}{q}-\frac{dq}{qq}\right)=\frac{p}{q}-\frac{pdq}{qq}+\frac{dp}{q}-\frac{dpdq}{qq},$$

unde ſi fractio ipſa $\frac{p}{q}$ ſubtrahatur, remanet eius differentiale

$$d.\frac{p}{q}=\frac{dp}{q}-\frac{pdq}{qq},$$ ob evaneſcentem terminum $\frac{dpdq}{qq}$.

Hinc

Hinc ergo erit $d.\frac{p}{q} = \frac{qdp - pdq}{qq}$, unde haec regula pro differentiatione cuiuſque fractionis enaſcitur. *A differentiali numeratoris per denominatorem multiplicato ſubtrahatur differentiale denominatoris per numeratorem multiplicatum, reſiduum dividatur per quadratum denominatoris, quotuſque erit differentiale fractionis quaeſitum.* Cuius regulae uſus per ſequentia exempla illuſtrabitur.

I. Si fuerit $y = \frac{x}{aa + xx}$, erit per hanc regulam

$$dy = \frac{(aa + xx)dx - 2xxdx}{(aa + xx)^2} = \frac{(aa - xx)dx}{(aa + xx)^2}.$$

II. Si fuerit $y = \frac{\surd(aa + xx)}{aa - xx}$; reperitur

$$dy = \frac{(aa - xx)xdx : \surd(aa + xx) + 2xdx\surd(aa + xx)}{(aa - xx)^2},$$

& facta reductione $dy = \frac{(3aa + xx)xdx}{(aa - xx)^2\surd(aa + xx)}$.

Saepenumero expedit ea regula uti, quae ſequitur ex formula priori $d.\frac{p}{q} = \frac{dp}{q} - \frac{pdq}{qq}$, qua differentiale fractionis aequale reperitur differentiali numeratoris per denominatorem diviſo, demto differentiali denominatoris per numeratorem multiplicato at per quadratum denominatoris diviſo. Ita

III. Si fuerit $y = \frac{aa - xx}{a^4 + aaxx + x^4}$, erit

$$dy = \frac{-2xdx}{a^4 + aaxx + x^4} - \frac{(aa - xx)(2aaxdx + 4x^3dx)}{(a^4 + aaxx + x^4)^2}$$

quae ad eundem denominatorem revocata praebet.

$$dy = \frac{-2xdx(2a^4 + 2aaxx - x^4)}{(a^4 + aaxx + x^4)^2}$$

165.

165. Haec iam ſufficiunt ad cuiuſque functionis rationalis ipſius x propoſitae differentiale inveſtigandum; ſi enim fuerit integra modus differentiandi iam ſupra eſt expoſitus. Sit igitur functio propoſita fracta, quae ſemper ad huiuſmodi formam reducetur:

$$y = \frac{A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c.}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 + \&c.}$$

Ponatur numerator $= p$ & denominator $= q$, ut fiat

$$y = \frac{p}{q}; \text{ eritque } dy = \frac{qdp - pdq}{qq}.$$ At cum fit:

$$p = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.$$
& $$q = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \&c.$$
erit $$dp = Bdx + 2Cxdx + 3Dx^2dx + 4Ex^3dx + \&c.$$
& $$dq = \beta dx + 2\gamma xdx + 3\delta x^2dx + 4\varepsilon x^3dx + \&c.$$
unde per multiplicationem obtinebitur:

$$\begin{array}{rlllll}
qdp = & \alpha Bdx & + 2\alpha Cxdx & + 3\alpha Dx^2dx & + 4\alpha Ex^3dx & + \&c. \\
 & & \beta B & + 2\beta C & + 3\beta D & + \&c. \\
 & & & \gamma B & + 2\gamma C & + \&c. \\
 & & & & \delta B & + \&c. \\
pdq = & \beta Adx & + \beta Bxdx & + \beta Cx^2dx & + \beta Dx^3dx & + \&c. \\
 & & 2\gamma A & + 2\gamma B & + 2\gamma C & + \&c. \\
 & & & 3\delta A & + 3\delta B & + \&c. \\
 & & & & 4\varepsilon A & + \&c.
\end{array}$$

Ex his itaque obtinebitur differentiale quaeſitum:

$$dy = \frac{\begin{array}{l}+\alpha B\\-\beta A\end{array} dx \begin{array}{l}+2\alpha C\\-2\gamma A\end{array} xdx \begin{array}{l}+3\alpha D\\+\beta C\\-\gamma B\\-3\delta A\end{array} xxdx \begin{array}{l}+4\alpha E\\+2\beta D\\-2\delta B\\-4\varepsilon A\end{array} x^3dx \begin{array}{l}+5\alpha F\\+3\beta E\\+\gamma D\\-\delta C\\-3\varepsilon B\\-5\zeta A\end{array} x^4dx \,\&c.}{(\alpha + \beta x + \gamma xx + \delta x^3 + \varepsilon x^4 + \zeta x^5 + \&c.)^2}$$

Quae

Quae expreſſio ad cuiuſvis functionis rationalis differentiale expedite inveniendum maxime eſt accommodata. Quemadmodum enim numerator differentialis ex coefficientibus numeratoris ac denominatoris functionis propoſitae combinatur, ex inſpectione mox intelligitur, denominator vero differentialis eſt quadratum denominatoris functionis propoſitae.

166. Si fractionis propoſitae vel numerator vel denominator vel uterque ex factoribus conſtet, multiplicatione actu inſtituta orietur quidem forma, qualem modo differentiavimus; attamen facilius pro his caſibus regula peculiaris formabitur. Sit igitur propoſita huiuſmodi fractio $y=\frac{pr}{q}$. Ponatur numerator $pr=P$, ut ſit $dP=pdr+rdp$. Atque ob $y=\frac{P}{q}$, erit $dy=\frac{qdP-Pdq}{qq}$ ſubſtitutis autem loco P & dP valoribus, habebitur:

I. Si fuerit $y=\frac{pr}{q}$; eius diff. $dy=\frac{pqdr+qrdp-prdq}{qq}$.

Si ſit $y=\frac{p}{qs}$, poſito denominatore $qs=Q$,

erit $dQ=qds+sdq$, & $dy=\frac{Qdp-pdQ}{qqss}$. Quare

II. Si fuerit $y=\frac{p}{qs}$, erit $dy=\frac{qsdp-pqds-psdq}{qqss}$.

Si fuerit $y=\frac{pr}{qs}$, ponatur $pr=P$ & $qs=Q$, ut habeatur $y=\frac{P}{Q}$, & $dy=\frac{QdP-PdQ}{QQ}$. Cum autem ſit

$$dP=pdr+rdp \quad \& \quad dQ=qds+sdq,$$

prodibit ſequens differentiatio:

III.

III. Si fuerit $y=\frac{pr}{qs}$,

erit $dy=\frac{pqsdr+qrsdp-pqrds-prsdq}{qqss}$,

ſeu $dy=\frac{rdp}{qs}+\frac{pdr}{qs}-\frac{prdq}{qqs}-\frac{prds}{qss}$.

Simili modo, ſi numerator ac denominator fractionis propoſitae plures habeant factores, differentialia eadem ratione inveſtigabuntur; neque ad hoc ampliori manuductione erit opus. Quamobrem quoque exempla huc pertinentia praetermitto, cum mox modus generalis has omnes differentiandi methodos particulares complectens afferetur.

167. Dantur autem caſus tam productorum quam fractionum, quibus differentiale commodius exprimi poteſt, quam per regulas generaliores hic expoſitas. Evenit hoc ſi factores, qui vel functionem ipſam, vel functionis numeratorem aut denominatorem conſtituunt, fuerint poteſtates.

Ponamus functionem differentiandam eſſe $y=p^mq^n$, ad cuius differentiale inveniendum ſit $p^m=P$ & $q^n=Q$, ut fiat $y=PQ$ & $dy=PdQ+QdP$. Cum autem ſit $dP=mp^{m-1}dp$ & $dQ=nq^{n-1}dq$, fiet his valoribus ſubſtitutis: $dy=np^mq^{n-1}dq+mp^{m-1}q^ndp=p^{m-1}q^{n-1}(npdq+mqdp)$; unde ſequens oritur regula:

I. Si fuerit $y=p^mq^n$;

erit $dy=p^{m-1}q^{n-1}(npdq+mqdp)$.

Simili modo ſi tres fuerint factores, differentiale invenietur, ac reperietur hoc modo expreſſum.

II. Si fuerit $y=p^mq^nr^k$;

erit $dy=p^{m-1}q^{n-1}r^{k-1}(mqrdp+nprdq+kpqdr)$.

168. Sin autem fuerit propoſita fractio, cuius vel numerator vel denominator habeat factorem, qui eſt poteſtas, regulae quoque particulares tradi poterunt. Sit primum propo-

ſita huiuſmodi fractio $y = \frac{p^m}{q}$, erit per regulam fractionibus inſervientem $dy = \frac{mp^{m-1}qdp - p^m dq}{qq}$, quod differentiale commodius ſic exprimetur.

I. Si fuerit $y = \frac{p^m}{q}$, erit $dy = \frac{p^{m-1}(mqdp - pdq)}{qq}$.

Sit iam $y = \frac{p}{q^n}$, fiet per eandem ſuperiorem regulam $dy = \frac{q^n dp - npq^{n-1}dq}{q^{2n}}$, cuius expreſſionis ſi numerator ac denominator per q^{n-1} dividatur, erit $dy = \frac{qdp - npdq}{q^{n+1}}$. Quamobrem.

II. Si fuerit $y = \frac{p}{q^n}$, erit $dy = \frac{qdp - npdq}{q^{n+1}}$.

Quod ſi vero proponatur $y = \frac{p^m}{q^n}$; invenietur
$$dy = \frac{mp^{m-1}q^n dp - np^m q^{n-1}dq}{q^{2n}}, \text{ quae reducitur ad}$$
$$dy = \frac{mp^{m-1}qdp - np^m dq}{q^{n+1}}. \quad \text{Quocirca}$$

III. Si fuerit $y = \frac{p^m}{q^n}$ erit $dy = \frac{p^{m-1}(mqdp - npdq)}{q^{n+1}}$.

Denique ſi propoſita fuerit huiuſmodi fractio $y = \frac{r}{p^m q^n}$, habebitur per regulam fractionum generalem
$$dy = \frac{p^m q^n dr - mp^{m-1}q^n rdp - np^m q^{n-1}rdq}{p^{2m}q^{2n}},$$
cuius expreſſionis cum numerator & denominator ſit diviſibilis per $p^{m-1}q^{n-1}$:

IV.

IV. Si fuerit $y = \frac{r}{p^m q^n}$;

erit $dy = \frac{pq\,dr - mqr\,dp - npr\,dq}{p^{m+1} q^{n+1}}$.

Si plures occurrant factores, huiuſmodi regulae ſpeciales, quas verbis exprimere ſuperfluum foret, facili negotio pro quovis caſu erui poterunt.

169. Regulae differentiandi quas hactenus expoſuimus tam late patent, ut nulla excogitari poſſit functio ipſius x algebraica, quae non earum ope differentiari queat. Si enim functio ipſius x fuerit rationalis, vel erit integra vel fracta, priori caſu §. 159. modum dedimus eiuſmodi functiones differentiandi, poſteriori vero caſu in §. 165. negotium abſolvimus. Simul vero etiam compendia, ſi factores involvantur, differentiationis exhibuimus. Deinde vero etiam quantitates irrationales cuiuſvis generis differentiare docuimus, quae quomodocunque functionem propoſitam afficiant, ſive ei per additionem, ſive per ſubtractionem ſive multiplicationem ſive diviſionem ſint implicatae, perpetuo ad caſus iam tractatos revocari poterunt. Intelligenda autem haec ſunt de functionibus explicitis; nam de implicitis, quarum natura per aequationem datur, infra, poſtquam functiones duarum pluriumve variabilium differentiare docuerimus, tractandi locus erit.

170. Si regulas hic traditas ſingulas perpendamus atque inter ſe conferamus, eas omnes ad unam maxime univerſalem reducere poterimus; quam autem infra demum rigida demonſtratione munire licebit; interim tamen & hoc loco non adeo difficile erit eius veritatem attendenti intueri. Functio quaecunque algebraica compoſita eſt ex partibus, quae vel additione vel ſubtractione vel multiplicatione vel diviſione inter ſe erunt complicatae; haeque partes erunt vel rationales vel irrationales. Vocemus ergo iſtas quantitates functionem quamvis conſtituentes eius partes. *Tum pro qualibet parte functio propoſi-*

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ta ſeorſim ita differentietur, quaſi ea pars ſola eſſet variabilis, reliquae vero partes omnes conſtantes. Quo facto ſingula iſta differentialia, quae ex ſingulis partibus modo deſcripto eliciuntur, in unam ſummam colligantur, ſicque obtinebitur differentiale functionis propoſitae. Huiuſque regulae ope omnes omnino functiones differentiari poterunt, nequidem tranſcendentibus exceptis, uti infra oſtendetur.

171. Ad regulam hanc illuſtrandam ponamus functionem y duabus conſtare partibus, ſive per additionem ſive ſubtractionem connexis, ita ut ſit $y = p \pm q$. Ponatur primo ſola pars p variabilis, altera q conſtans erit differentiale $= dp$, deinde ponatur altera pars $\pm q$ ſola variabilis, altera vero p conſtans, eritque differentiale $= \pm dq$. Atque ex his differentialibus differentiale quaeſitum ita componetur, ut ſit $dy = dp \pm dq$, omnino uti idem iam ſupra invenimus. Hinc vero ſimul liquet, ſi functio pluribus conſtet partibus, ſive invicem additis ſive ſubtractis, nempe $y = p \pm q \pm r \pm s$, ope huius regulae inventum iri $dy = dp \pm dq \pm dr \pm ds$, plane uti & ſuperior regula docebat.

172. Si partes ſint in ſe invicem multiplicatae, ita ut ſit $y = pq$, manifeſtum eſt poſita ſola parte p variabili, ſore differentiale $= qdp$; at ſi altera pars q ſola variabilis ſtatuatur, erit differentiale $= pdq$. Addantur ergo haec duo differentialia invicem, atque prodibit differentiale quaeſitum $dy = qdp + pdq$, quemadmodum ex iam allatis conſtat. Si plures fuerint partes per multiplicationem connexae, ſcilicet $y = pqrs$, ſi ſucceſſive unaquaeque ſola variabilis ſtatuatur, orientur iſta differentialia $qrsdp$, $prsdq$, $pqsdr$, & $pqrds$, quorum ſumma dabit differentiale quaeſitum, nempe

$$dy = qrsdp + prsdq + pqsdr + pqrds,$$

prorſus uti iam ante invenimus. Differentiale ergo ex totidem partibus componitur, ſive partes functionem conſtituentes ſint invicem additae ſubtractaeve, ſive in ſe invicem multiplicatae

173. Si partes functionem formantes per diviſionem ſint connexae, nempe $y = \frac{p}{q}$ ponatur ſecundum regulam primum ſola pars p variabilis, eritque ob q conſtans differentiale $= \frac{dp}{q}$; deinde ponatur ſola pars q variabilis ob $y = pq^{-1}$, erit differentiale $= -\frac{pdq}{qq}$, quae duo differentialia collecta dabunt differentiale functionis propoſitae

$$dy = \frac{dp}{q} - \frac{pdq}{qq} = \frac{qdp - pdq}{qq},$$

ſicut iam ſupra invenimus. Simili modo ſi functio propoſita ſit $y = \frac{pq}{rs}$, ponendo ſucceſſive ſingulas partes ſolas p, q, r & s variabiles, prodibunt ſequentia differentialia:

$$\frac{qdp}{rs};\ \frac{pdq}{rs};\ \frac{-pqdr}{rrs};\ \& \frac{-pqds}{rss},\quad \text{unde fit}$$

$$dy = \frac{qrsdp + prsdq - pqsdr - pqrds}{rrss}$$

174. Dummodo ergo ſingulae partes, ex quibus functio componitur, ita fuerint comparatae; ut earum differentialia exhiberi queant, ſimul quoque totius functionis differentiale inveniri poterit. Quodſi igitur partes fuerint functiones rationales, tum earum differentialia non ſolum ope praeceptorum ante iam datorum inveniuntur, ſed ea quoque ex hac ipſa regula generali erui poterunt: ſin autem partes fuerint irrationales, quia irrationalitas ad poteſtates, quarum exponentes ſunt numeri fracti, reducitur, eae per differentiationem poteſtatum, qua eſt $d.x^n = nx^{n-1}dx$ differentiabuntur. Atque, ex eodem fonte haurietur quoque differentiatio eiuſmodi formularum irrationalium, quae alias inſuper expreſſiones ſurdas involvunt

volvunt. Unde patet ſi cum regula generali hic data, infra vero demonſtranda, coniungatur regula differentiandi poteſtates, tum omnium omnino functionum algebraicarum differentialia exhiberi poſſe.

175. Ex his omnibus iam dilucide ſequitur, ſi y fuerit functio quaecunque ipſius x, differentiale eius dy huiuſmodi habiturum eſſe formam $dy = pdx$, in qua valor ipſius p per praecepta hic expoſita ſemper aſſignari queat. Erit autem p functio ipſius x quoque algebraica, cum in eius determinationem nullae aliae operationes ingrediantur, niſi conſuetae, quibus functiones algebraicae conſtitui ſolent. Hancobrem ſi y fuerit functio algebraica ipſius x, erit quoque $\frac{dy}{dx}$ functio algebraica ipſius x. Atque ſi z fuerit etiam functio algebraica ipſius x, ita ut ſit $dz = qdx$, ob q functionem algebraicam ipſius x, erit quoque $\frac{dz}{dy}$ functio algebraica ipſius x, quippe quae eſt $= \frac{q}{p}$ Quare ſi huiuſmodi formulae $\frac{dz}{dy}$ in expreſſionem cetera algebraicam ingrediantur, eae non impedient, quominus ea expreſſio ſit algebraica, dummodo y & z fuerint functiones algebraicae.

176. Poterimus autem hoc ratiocinium extendere ad differentialia ſecunda & ſuperiorum ordinum. Si enim manente y functione algebraica ipſius x, fuerit $dy = pdx$, atque $dp = qdx$; erit ſumto differentiali dx conſtante, $ddy = qdx^2$ uti ſupra vidimus. Cum igitur ob rationes ante allegatas ſit quoque q functio algebraica ipſius x, erit quoque $\frac{ddy}{dx^2}$ non ſolum quantitas finita, ſed etiam functio algebraica ipſius x, dummodo y fuerit eiuſmodi functio. Simili modo perſpicietur, fore $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, &c. functiones al-

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gebraicas ipſius x, modo y fuerit talis; atque ſi z ſit quoque functio algebraica ipſius x, omnes expreſſiones finitae; quae ex differentialibus cuiuſvis ordinis ipſarum y, z, & ex dx componuntur cuiuſmodi ſunt $\frac{ddy}{ddz}$; $\frac{d^3y}{dzddy}$; $\frac{dxd^4y}{dy^3ddz}$; &c. ſimul erunt functiones algebraicae ipſius x.

177. Cum igitur nunc methodus ſit tradita cuiuſque functionis ipſius x algebraicae differentiale primum inveniendi, eadem methodo poterimus quoque differentialia ſecunda altiorumque ordinum inveſtigare. Si enim y fuerit functio quaecunque algebraica ipſius x, ex eius differentiatione $dy = pdx$ innoteſcet valor ipſius p. Qui ſi denuo differentietur atque reperiatur $dp = qdx$, erit $ddy = qdx^2$, poſito dx conſtante, ſicque definietur differentiale ſecundum. Differentiando porro q, ut ſit $dq = rdx$, habebitur differentiale tertium $d^3y = rdx^3$; ſicque ulterius differentialia altiorum ordinum indagabuntur; quoniam quantitates p, q, r, &c. omnes ſunt functiones ipſius x algebraicae, ad quas differentiandas praecepta data ſufficiunt. Hoc ergo efficietur continua differentiatione; omiſſis enim dx, in differentiatione ipſius y, prodibit valor ipſius $\frac{dy}{dx} = p$, qui denuo differentiatus ac diviſus per dx, quod fit dum ubique differentiale dx omittatur, dabit valorem ipſius $q = \frac{ddy}{dx^2}$. Simili modo porro invenitur $r = \frac{d^3y}{dx^3}$ &c.

I. Sit $y = \frac{aa}{aa + xx}$ cuius differentialia tam prima quam ſequentium ordinum requiruntur.

Primum ergo differentiando ſimulque per dx dividendo erit $\frac{dy}{dx} = \frac{-2aax}{(aa + xx)^2}$, hincque porro

$$\frac{ddy}{dx^2} = \frac{-2a^3 + 6aaxx}{(aa+xx)^3}$$

$$\frac{d^3y}{dx^3} = \frac{24a^4x - 24aax^3}{(aa+xx)^4}$$

$$\frac{d^4y}{dx^4} = \frac{24a^6 - 240a^4xx + 120aax^4}{(aa+xx)^5}$$

$$\frac{d^5y}{dx^5} = \frac{-720a^6x + 2400a^4x^3 - 720aax^5}{(aa+xx)^6}$$

&c.

II. Sit $y = \frac{1}{\sqrt{(1-xx)}}$, eruntque differentialia primum & ſequentia:

$$\frac{dy}{dx} = \frac{x}{(1-xx)^{\frac{3}{2}}}$$

$$\frac{ddy}{dx^2} = \frac{1+2xx}{(1-xx)^{\frac{5}{2}}}$$

$$\frac{d^3y}{dx^3} = \frac{9x+6x^3}{(1-xx)^{\frac{7}{2}}}$$

$$\frac{d^4y}{dx^4} = \frac{9+72x^2+24x^4}{(1-xx)^{\frac{9}{2}}}$$

$$\frac{d^5y}{dx^5} = \frac{225x+600x^3+120x^5}{(1-xx)^{\frac{11}{2}}}$$

$$\frac{d^6y}{dx^6} = \frac{225+4050x^2+5400x^4+720x^6}{(1-xx)^{\frac{13}{2}}}$$

&c.

Haec

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Haec Differentialia facile ulterius continuantur; interim tamen lex, qua termini eorum progrediuntur, non cito patet. Coefficiens quidem ſupremarum ipſius x poteſtatum ſemper eſt productum numerorum naturalium ab 1 uſque ad ordinem differentialis, quod quaeritur. Interim ſi has formas ulterius continuemus atque perpendamus, deprehendemus fore generaliter, ſi $y = \frac{1}{\sqrt{(1-xx)}}$,

$$\frac{d^n y}{dx^n} = \frac{1.2.3 \ldots\ldots n}{(1-xx)^{n+\frac{1}{2}}} \Big(x^n + \frac{1}{2} \cdot \frac{n(n-1)}{1.\ 2.} x^{n-2} + \frac{1.3}{2.4} \cdot \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4} x^{n-4} + \frac{1.3.5}{2.4.6} \cdot \frac{n(n-1)\ldots(n-5)}{1.\ 2\ldots 6} x^{n-6} + \frac{1.3.5.7}{2.4.6.8} \cdot \frac{n(n-1)\ldots\ldots(n-7)}{1.\ 2 \ldots\ldots 8} x^{n-8} + \&c. \Big)$$

Huiuſmodi ergo exempla non ſolum inſerviunt ad habitum in differentiationis negotio acquirendum, ſed etiam leges, quae in differentialibus omnium ordinum obſervantur, per ſe ſunt notatu digniſſimae, atque ad alias inventiones deducere poſſunt.

CAPUT VI.

DE DIFFERENTIATIONE FUNCTIONUM TRANSCENDENTIUM.

178.

Praeter infinita quantitatum tranſcendentium ſeu non algebraicarum genera, quae calculus integralis ſuppeditabit, in Introductione ad analyſin infinitorum ad cognitionem aliquot huiuſmodi quantitatum magis uſitatarum nobis pervenire licuit, quas doctrina de logarithmis & arcubus circularibus ſuggeſſerat. Quoniam igitur harum quantitatum naturam tam dilucide expoſuimus, ut fere eadem facilitate atque quantitates algebraicae in calculo tractari queant, earum quoque differentialia in hoc capite inveſtigabimus, quo earum indoles ac proprietates clarius perſpiciantur; hocque pacto aditus ad calculum integralem, qui quantitatum tranſcendentium eſt fons proprius, pateſiat.

179. Primum igitur occurrunt quantitates logarithmicae, ſeu eiuſmodi functiones ipſius x, quae praeter expreſſiones algebraicas quoque logarithmum ipſius x, ſeu cuiuſvis ipſius functionis involvunt. Ad quas differentiandas, cum quantitates algebraicae nullum negotium amplius faceſſant, omnis difficultas in inveniendo differentiali logarithmi cuiuſque ipſius x functionis erit poſita. Quia vero logarithmorum plurima dantur genera diverſa, quae tamen inter ſe conſtantes tenent rationes, hic logarithmos hyperbolicos potiſſimum contemplabimur, cum ex iis omnes reliqui logarithmi facile formentur. Si enim functionis p logarithmus hyperbolicus fuerit $= lp$, tum eiuſdem functionis p logarithmus ex alio canone deſumtus erit $= mlp$, denotante m numerum, quo relatio huius

huius logarithmorum canonis ad hyperbolicos exprimitur. Hanc ob cauſam lp perpetuo hic deſignabit logarithmum hyperbolicum quantitatis p.

180. Quaeramus ergo differentiale logarithmi hyperbolici quantitatis x, ponaturque $y = lx$, ita ut differentialis dy valor definiri debeat. Ponatur $x + dx$ loco x, ſicque tranſibit y in $y^{I} = y + dy$; quare habebitur

$$y + dy = l(x + dx) \ \& \ dy = l(x + dx) - lx = l(1 + \frac{dx}{x}).$$

At iam ſupra logarithmum hyperbolicum huiuſmodi expreſſionis $1 + z$ ita per ſeriem infinitam expreſſimus, ut eſſet

$$l(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \&c.$$

Poſito ergo $\frac{dx}{x}$ pro z, obtinebimus:

$$dy = \frac{dx}{x} - \frac{dx^2}{2x^2} + \frac{dx^3}{3x^3} - \&c.$$

Cum igitur huius ſeriei omnes termini prae primo evaneſcant, erit $d.\ lx = dy = \frac{dx}{x}$. Unde alius cuiuſcunque logarithmi, cuius ad hyperbolicum ratio eſt ut $n:1$, differentiale erit $= \frac{ndx}{x}$.

181. Si igitur cuiuſque ipſius x functionis p logarithmus lp proponatur, eodem ratiocinio reperietur eius differentiale eſſe $= \frac{dp}{p}$, unde ad logarithmorum differentialia invenienda haec habetur regula. *Quantitatis* p, *cuius logarithmus proponitur*, *ſumatur differentiale*, *hocque per ipſam quantitatem* p *diviſum dabit differentiale logarithmi quaeſitum*. Se-

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quitur haec eadem regula quoque ex forma $\frac{p^{\omega}-1^{\omega}}{o}$, ad quam ſuperiori libro logarithmum ipſius p reduximus. Sit $\omega = o$, & cum ſit

$lp = \frac{p^{\omega}-1}{\omega}$: erit $d.lp = d.\frac{1}{\omega}p^{\omega} = p^{\omega-1}dp = \frac{dp}{p}$ ob $\omega = o$.

Notandum autem eſt $\frac{dp}{p}$ eſſe differentiale logarithmi hyperbolici ipſius p; ita ut, ſi logarithmus vulgaris ipſius p proponeretur, differentiale illud $\frac{dp}{p}$ multiplicari deberet per hunc numerum 0,43429448 &c.

182. Ope huius ergo regulae, cuiuſcunque functionis ipſius x logarithmus proponatur, eius differentiale facillime inveniri poterit, quemadmodum ex ſequentibus exemplis perſpicietur:

I. Si ſit $y = lx$; erit $dy = \frac{dx}{x}$.

II. Si ſit $y = lx^{n}$; ponatur $x^{n} = p$, ut ſit $y = lp$, eritque $dy = \frac{dp}{p}$. At eſt $dp = nx^{n-1}dx$, unde ſit $dy = \frac{ndx}{x}$.

Idem quoque ex logarithmorum natura colligitur; cum enim ſit $lx^{n} = nlx$, erit $d.lx^{n} = nd.lx = \frac{ndx}{x}$.

III. Si ſit $y = l(1+xx)$, erit $dy = \frac{2xdx}{1+xx}$.

IV. Si ſit $y = l\frac{1}{\sqrt{(1-xx)}}$; quia erit $y = -l\sqrt{(1-xx)}$ $= -\frac{1}{2}l(1-xx)$, invenitur $dy = \frac{xdx}{1-xx}$.

V.

V. Si ſit $y = l\frac{x}{\surd(1+xx)}$, ob $y = lx - \frac{1}{2}l(1+xx)$,

fiet $dy = \frac{dx}{x} - \frac{xdx}{1+xx} = \frac{dx}{x(1+xx)}$.

VI. Si ſit $y = l(x + \surd(1+xx))$, fiet

$$dy = \frac{dx + xdx : \surd(1+xx)}{x + \surd(1+xx)} = \frac{xdx + dx\surd(1+xx)}{(x+\surd(1+xx))\surd(1+xx)}$$

cuius fractionis cum numerator ac denominator per $x + \surd(1+xx)$ ſit diviſibilis fiet $dy = \frac{dx}{\surd(1+xx)}$.

VII. Si ſit $y = \frac{1}{\surd -1}\, l(x\surd -1 + \surd(1-xx))$, ponatur $x\surd -1 = z$. Atque ob $y = \frac{1}{\surd -1}\, l(z + \surd(1+zz))$, erit per praecedens $dy = \frac{1}{\surd -1}\, dz : \surd(1+zz)$.

Quare, ob $dz = dx\surd -1$, fiet $dy = \frac{dx}{\surd(1-xx)}$.

Quamvis ergo logarithmus propoſitus imaginaria involvat, tamen eius differentiale fit reale.

183. Si quantitas, cuius logarithmus proponitur, habeat factores, tum ipſe logarithmus in plures alios reſolvetur hoc modo: Si proponatur $y = lpqrs$, quia erit $y = lp + lq + lr + ls$, erit $dy = \frac{dp}{p} + \frac{dq}{q} + \frac{dr}{r} + \frac{ds}{s}$.
Haec reſolutio pariter locum habet, ſi illa quantitas, cuius logarithmus differentiari debet, fuerit fractio. Sit enim $y = l\frac{pq}{rs}$, ob $y = lp + lq - lr - ls$, erit $dy = \frac{dp}{p} + \frac{dq}{q} - \frac{dr}{r} - \frac{ds}{s}$.
Neque etiam poteſtates difficultatem movebunt, ſi enim fue-

fuerit $y = l\frac{p^m q^n}{r^\mu s^\nu}$, ob $y = mlp + nlq - \mu lr - \nu ls$,

erit $dy = \frac{mdp}{p} + \frac{ndq}{q} - \frac{\mu dr}{r} - \frac{\nu ds}{s}$.

I. Si fuerit $y = l(a+x)(b+x)(c+x)$, quia erit $y = l(a+x) + l(b+x) + l(c+x)$, fiet differentiale quaeſitum
$$dy = \frac{dx}{a+x} + \frac{dx}{b+x} + \frac{dx}{c+x}.$$

II. Si fuerit $y = \frac{1}{2}l\frac{1+x}{1-x}$, erit $y = \frac{1}{2}l(1+x) - \frac{1}{2}l(1-x)$,

hincque $dy = \frac{\frac{1}{2}dx}{1+x} + \frac{\frac{1}{2}dx}{1-x} = \frac{dx}{1-xx}$.

III. Si ſit $y = \frac{1}{2}l\frac{\surd(1+xx)+x}{\surd(1+xx)-x}$, ob $y = \frac{1}{2}l(\surd(1+xx)+x)$ $-\frac{1}{2}l(\surd(1+xx)-x)$, erit $dy = \frac{\frac{1}{2}dx}{\surd(1+xx)} + \frac{\frac{1}{2}dx}{\surd(1+xx)} =$ $\frac{dx}{\surd(1+xx)}$. Hoc idem facilius invenitur, ſi in fractione $\frac{\surd(1+xx)+x}{\surd(1+xx)-x}$, irrationalitas in denominatore tollatur multiplicando numeratorem ac denominatorem per $\surd(1+xx)+x$, prodibit enim
$$y = \frac{1}{2}l(\surd(1+xx)+x)^2 = l(\surd(1+xx)+x),$$
cuius differentiale ante vidimus eſſe $dy = \frac{dx}{\surd(1+xx)}$.

IV. Si ſit $y = l\frac{\surd(1+x)+\surd(1-x)}{\surd(1+x)-\surd(1-x)}$. Ponatur huius fractionis numerator $\surd(1+x)+\surd(1-x) = p$ & denominator $\surd(1+x)-\surd(1-x) = q$, erit $y = l\frac{p}{q} = lp - lq$,

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& $dy = \frac{dp}{p} - \frac{dq}{q}$. Eſt vero $dp = \frac{dx}{2\surd(1+x)} - \frac{dx}{2\surd(1-x)} = \frac{-dx}{2\surd(1-xx)}(\surd(1+x) - \surd(1-x)) = \frac{-qdx}{2\surd(1-xx)}$; & $dq = \frac{dx}{2\surd(1+x)} + \frac{dx}{2\surd(1-x)} = \frac{pdx}{2\surd(1-xx)}$. Hinc fiet $\frac{dp}{p} - \frac{dq}{q} = \frac{-qdx}{2p\surd(1-xx)} - \frac{pdx}{2q\surd(1-xx)} = \frac{-(pp+qq)dx}{2pq\surd(1-xx)}$. At eſt $pp + qq = 4$ & $pq = 2x$, unde erit $dy = -\frac{dx}{x\surd(1-xx)}$. Hoc autem differentiale facilius invenietur, ſi logarithmus propoſitus ita transformetur,

$$y = l\,\frac{1+\surd(1-xx)}{x} = l\left(\frac{1}{x} + \surd\left(\frac{1}{xx} - 1\right)\right).$$

Poſito enim $\frac{1}{x} + \surd\left(\frac{1}{xx} - 1\right) = p$, erit

$$dp = \frac{-dx}{xx} - \frac{dx}{x^3\surd\left(\frac{1}{xx}-1\right)} = \frac{-dx}{xx} - \frac{dx}{xx\surd(1-xx)}$$

$$= \frac{-dx(1+\surd(1-xx))}{xx\surd(1-xx)},$$ ideoque, ob $p = \frac{1+\surd(1-xx)}{x}$,

erit $dy = \frac{dp}{p} = \frac{-dx}{x\surd(1-xx)}$ ut ante.

184. Cum igitur logarithmorum differentialia prima, ſi per dx dividantur, ſint quantitates algebraicae, differentialia ſecunda ac ſequentium ordinum per praecepta praecedentis capitis facile inveniuntur, ſi quidem differentiale dx aſſumatur conſtans. Sic poſito

$y =$

$$y = lx\,, \qquad \text{erit}$$

$$dy = \frac{dx}{x}\,, \quad \& \quad \frac{dy}{dx} = \frac{1}{x}$$

$$ddy = \frac{-dx^2}{x^2}\,, \quad \& \quad \frac{ddy}{dx^2} = \frac{-1}{x^2}$$

$$d^3y = \frac{2dx^3}{x^3}\,, \quad \& \quad \frac{d^3y}{dx^3} = \frac{2}{x^3}$$

$$d^4y = \frac{-6dx^4}{x^4}\,, \& \quad \frac{d^4y}{dx^4} = \frac{-6}{x^4}$$

Atque ſi p fuerit quantitas algebraica, ſitque $y = lp$, etiamſi y non ſit quantitas algebraica, tamen $\frac{dy}{dx}$; $\frac{ddy}{dx^2}$; $\frac{d^3y}{dx^3}$; &c. erunt functiones algebraicae ipſius x.

185. Expoſita logarithmorum differentiatione, functiones, quae ex algebraicis ac logarithmis ſunt permixtae, facile differentiabuntur, perinde atque eae, quae ex logarithmis ſolis componuntur; uti ex ſequentibus exemplis fiet perſpicuum.

I. Si ſit $y = (lx)^2$, ponatur $lx = p$, atque ob $y = p^2$ erit $dy = 2pdp$; verum $dp = \frac{dx}{x}$; ideoque erit $dy = \frac{2dx}{x}\,lx$.

II. Simili modo ſi ſit $y = (lx)^n$, erit $dy = \frac{ndx}{x}(lx)^{n-1}$, unde, ſi ſit $y = \sqrt{lx}$, ob $n = \frac{1}{2}$, erit $dy = \frac{dx}{2x\sqrt{lx}}$.

III. Atque ſi p fuerit functio quaecunque ipſius x, ponaturque $y = (lp)^n$, erit $dy = \frac{ndp}{p}(lp)^{n-1}$. Quare cum differentiale dp per praecedentia aſſignari poſſit, erit quoque differentiale ipſius y cognitum.

IV. Si ſit $y = lp \,.\, lq$, fuerintque p & q functiones quaecun-

cunque ipſius x, per regulam factorum ſupra datam erit

$$dy = \frac{dp}{p} lq + \frac{dq}{q} lp.$$

V. Si ſit $y = xlx$; erit per eandem regulam $dy = dxlx + \frac{xdx}{x} = dxlx + dx$.

VI. Si ſit $y = x^m lx - \frac{1}{m} x^m$, differentiatione ſecundum partes inſtituta, reperietur $d.x^m lx = mx^{m-1} dxlx + x^{m-1} dx$, & $d.\frac{1}{m} x^m = x^{m-1} dx$, unde erit $dy = mx^{m-1} dxlx$.

VII. Si ſit $y = x^m (lx)^n$, fiet $dy = mx^{m-1} dx (lx)^n + nx^{m-1} dx (lx)^{n-1}$.

VIII. Si logarithmi logarithmorum occurrant, uti ſi fuerit $y = llx$, ponatur $lx = p$, erit $y = lp$, & $dy = \frac{dp}{p}$; at eſt $dp = \frac{dx}{x}$; unde fiet $dy = \frac{dx}{xlx}$.

IX. Atque ſi fuerit $y = lllx$, ſi ſtatuatur $lx = p$, fiet $y = llp$, eritque per exemplum praecedens $dy = \frac{dp}{plp}$; at eſt $dp = \frac{dx}{x}$, quibus valoribus ſubſtitutis habebitur $dy = \frac{dx}{xlx.llx}$.

186. Expoſita logarithmorum differentiatione, progrediamur ad quantitates exponentiales, ſeu eiuſmodi poteſtates, quarum exponentes ſint variabiles. Huiuſmodi autem ipſius x functionum differentialia per logarithmorum differentiationem inveniri poſſunt hoc modo. Quaeratur differentiale ipſius a^x, ad quod inveſtigandum ponatur $y = a^x$, eritque logarithmis ſumendis $ly = xla$. Sumantur iam differentialia, atque obti-

nebitur $\frac{dy}{y} = dxla$; unde fit $dy = ydxla$, cum autem ſit $y = a^x$, erit $dy = a^x dxla$, quod eſt differentiale ipſius a^x. Simili modo, ſi ſit p functio quaecunque ipſius x, huius quantitatis exponentialis a^p differentiale erit $= a^p dpla$.

187. Hoc idem autem differentiale immediate ex natura quantitatum exponentialium in introductione expoſita deduci poteſt. Sit enim propoſita a^p, denotante p functionem quamcunque ipſius x, quae, poſito $x + dx$ loco x, abeat in $p + dp$. Unde ſi ponatur $y = a^p$, ſi x abeat in $x + dx$, erit $y + dy = a^{p+dp}$, ideoque $dy = a^{p+dp} - a^p = a^p(a^{dp} - 1)$. Oſtendimus autem ſupra, quamvis quantitatem exponentialem a^z, per huiuſmodi ſeriem exprimi $1 + zla + \frac{z^2(la)^2}{2} + \frac{z^3(la)^3}{6} + \&c.$ unde erit $a^{dp} = 1 + dpla + \frac{dp^2(la)^2}{2} + \&c.$, & $a^{dp} - 1 = dpla$, quia ſequentes termini prae $dpla$ omnes evaneſcunt. Conſequenter erit $dy = d.a^p = a^p dpla$. *Quare quantitatis exponentialis* ap *differentiale erit productum ex ipſa quantitate exponentiali, exponentis differentiali* dp, *& logarithmo quantitatis conſtantis* a, *quae ad exponentem variabilem eſt evecta.*

188. Si igitur e ſit numerus, cuius logarithmus hyperbolicus eſt $= 1$, ut ſit $le = 1$, erit quantitatis e^x differentiale $= e^x dx$. Atque ſi dx ſumatur conſtans, erit huius differentiale $= e^x dx^2$, quod eſt differentiale ſecundum ipſius e^x. Simili modo differentiale tertium erit $= e^x dx^3$. Quare ſi ſit

$$y = e^{nx}, \quad \text{erit} \quad \frac{dy}{dx} = ne^{nx}, \quad \& \quad \frac{ddy}{dx^2} = n^2 e^{nx}$$

$$\text{porroque} \quad \frac{d^3y}{dx^3} = n^3 e^{nx}; \quad \frac{d^4y}{dx^4} = n^4 e^{nx}; \quad \&c.$$

Unde patet ipſius e^{nx} differentialia primum, ſecundum & reli-

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reliqua ſequentia conſtituere progreſſionem geometricam: eritque ergo differentiale ordinis m ipſius $e^{nx} = y$, nempe $\frac{d^m y}{dx^m} = n^m e^{nx}$; hincque igitur $\frac{d^m y}{y dx^m}$ quantitas conſtans n^m.

189. Si ipſa quantitas, quae elevatur, fuerit variabilis, eius differentiale ſimili modo inveſtigabitur. Sint p & q functiones quaecunque ipſius x, ac proponatur quantitas exponentialis $y = p^q$. Sumtis logarithmis erit $ly = qlp$, quibus differentiatis erit $\frac{dy}{y} = dqlp + \frac{qdp}{p}$, unde fit

$$dy = ydqlp + \frac{yqdp}{p} = p^q dqlp + qp^{q-1}dp,$$ ob $y = p^q$. Hoc ergo differentiale conſtat duobus membris, quorum prius $p^q dqlp$ oritur, ſi quantitas propoſita p^q ita differentietur, quaſi p eſſet quantitas conſtans, ſolusque exponens q variabilis: alterum vero membrum $qp^{q-1}dp$ oritur, ſi in quantitate propoſita p^q exponens q tanquam conſtans ſpectetur, ſolaque quantitas p, quaſi eſſet variabilis, tractetur. Hocque ergo differentiale per regulam generalem differentiandi ſupra traditam inveniri potuiſſet.

190. Eiuſdem vero expreſſionis p^q differentiale quoque ex natura quantitatum exponentialium erui poteſt hoc modo: ſit $y = p^q$, eritque, loco x poſito $x + dx$, utique $y + dy = (p + dp)^{q+dq}$, quae expreſſio, ſi more ſolito in ſeriem reſolvatur, fiet

$$y + dy = p^{q+dq} + (q + dq)\, p^{q+dq-1} dp + \frac{(q + dq)\,(q + dq - 1)}{1 \cdot 2} p^{q+dq-2} dp^2 + \&c.$$

ideoque

$$dy = p^{q+dq} - p^q + (q + dq) p^{q+dq-1} dp,$$

ſequentes enim termini, qui altiores ipſius dp poteſtates involvunt, prae $(q + dq) p^{q+dq-1} dp$ evaneſcunt. At eſt

$p^{q+dq}-p^q = p^q(p^{dq}-1) = p^q(1+dqlp+\frac{dq^2(lp)^2}{2}+\&c.-1)$ $= p^q dqlp$. In altero vero termino $(q+dq)p^{q+dq-1}dp$ ſi loco $q+dq$ ſcribamus q, orietur $qp^{q-1}dp$, ideoque differentiale erit ut ante $dy = p^q dqlp + qp^{q-1}dp$.

191. Facilius vero hoc idem differentiale ex natura quantitatum exponentialium inveſtigabitur, hoc modo: Cum, ſumto e pro numero, cuius logarithmus hyperbolicus eſt $=1$, ſit $p^q = e^{qlp}$, utriuſque enim logarithmus eſt idem qlp; erit $y = e^{qlp}$. Quare, cum nunc quantitas elevata e ſit conſtans, erit $dy = e^{qlp}\left(dqlp + \frac{qdp}{p}\right)$, uti ante oſtendimus in regula §. 187. data. Reſtituatur igitur p^q loco e^{qlp}, fietque

$$dy = p^q dqlp + p^q qdp : p = p^q dqlp + qp^{q-1}dp.$$

Si igitur fuerit $y = x^x$, erit $dy = x^x dxlx + x^x dx$; atque hinc quoque eius ulteriora differentialia definientur: reperietur enim:

$$\frac{ddy}{dx^2} = x^x\left(\frac{1}{x} + (1+lx)^2\right)$$

$$\frac{d^3y}{dx^3} = x^x\left((1+lx)^3 + \frac{3(1+lx)}{x} - \frac{1}{xx}\right)$$

&c.

192. Inter differentialia huiuſmodi functionum, quae quantitates exponentiales complectuntur, imprimis ſunt notanda ſequentia exempla, quae ex differentiatione formulae $e^x p$ originem habent; eſt autem

$$d.\,e^x p = e^x dp + e^x pdx = e^x(dp + pdx).$$

I. Si ſit $y = e^x x^n$; erit $dy = e^x nx^{n-1}dx + e^x x^n dx$
ſeu $dy = e^x dx(nx^{n-1} + x^n)$

II. Si ſit $y = e^x(x-1)$
Erit $dy = e^x xdx$.

III.

III. Si ſit $y = e^x(x^2 - 2x + 2)$
Erit $dy = e^x xxdx$.

IV. Si ſit $y = e^x(x^3 - 3x^2 + 6x - 6)$
Erit $dy = e^x x^3 dx$.

193. Si ipſi exponentes fuerint denuo quantitates exponentiales, differentiatio ſecundum eadem praecepta inſtituetur. Sic ſi haec quantitas e^{e^x} differentiari debeat, ſtatuatur $e^x = p$, ut ſit $y = e^{e^x} = e^p$, erit $dy = e^p dp$; at eſt $dp = e^x dx$, unde ſi fuerit

$$y = e^{e^x}\;;\quad \text{erit}\quad dy = e^{e^x} e^x dx,$$

atque ſi ſit $y = e^{e^{e^x}}$; erit $dy = e^{e^{e^x}} e^{e^x} e^x dx$.

Quod ſi vero fuerit $y = p^{q^r}$, ſtatuatur $q^r = z$, erit $dy = p^z dzlp + zp^{z-1}dp$, at $dz = q^r drlq + rq^{r-1}dq$, unde $dy = p^z q^r drlp.lq + p^z rq^{r-1}dqlp + p^z q^r dp : p$.

Quare ſi ſit:

$$y = p^{q^r}, \quad \text{erit}\quad dy = p^{q^r} q^r \left(drlp.lq + \frac{rdqlp}{q} + \frac{dp}{p}\right).$$

Hoc ergo modo, quaecunque occurrat quantitas exponentialis, eius differentiale inveniri poterit.

194. Pergamus ergo ad quantitates tranſcendentes, ad quarum cognitionem conſideratio arcuum circularium nos ſupra deduxit. Sit igitur in circulo, cuius radium conſtanter ponimus unitati aequalem, propoſitus arcus, cuius ſinus ſit

=

$= x$, quem arcum hoc modo exprimamus A ſin x, huiuſque arcus differentiale inveſtigemus, ſeu incrementum quod accipit, ſi ſinus x differentiali ſuo dx augeatur. Hoc autem ex differentiatione logarithmorum praeſtari poterit, quia in introductione oſtendimus hanc expreſſionem A ſin x reduci poſſe ad hanc logarithmicam:

$\frac{1}{\surd -1}\, l(\surd(1-xx)+x\surd -1)$. Poſito ergo $y = \text{A ſin}\, x$, erit quoque $y = \frac{1}{\surd -1}\, l(\surd(1-xx)+x\surd -1)$; quae differentiata dat

$$dy = \frac{\frac{1}{\surd -1}\left(\frac{-x\,dx}{\surd(1-xx)} + dx\surd -1\right)}{\surd(1-xx)+x\surd -1} = \frac{dx(x\surd -1+\surd(1-xx))}{(\surd(1-xx)+x\surd -1)\surd(1-xx)}$$

unde fit $dy = \frac{dx}{\surd(1-xx)}$.

195. Iſtud arcus circularis differentiale etiam hoc modo facilius ſine logarithmorum ſubſidio inveniri poteſt. Si enim ſit $y = \text{A ſin}\, x$, erit x ſinus arcus y, ſeu $x = \text{ſin}\, y$. Cum igitur, poſito $x + dx$ loco x, abeat y in $y + dy$, fiet $x + dx = \text{ſin}\,(y + dy)$. At quia eſt

$\text{ſin}\,(a + b) = \text{ſin}\, a.\, \text{coſ}\, b + \text{coſ}\, a.\, \text{ſin}\, b$, erit

$\text{ſin}\,(y + dy) = \text{ſin}\, y.\, \text{coſ}\, dy + \text{coſ}\, y.\, \text{ſin}\, dy$:

arcus autem evaneſcentis dy ſinus ipſi illi arcui dy, eiuſque coſinus ſinui toti aequatur, hanc ob rem fiet

$\text{ſin}(y + dy) = \text{ſin}\, y + dy\, \text{coſ}\, y$, ideoque $x + dx = \text{ſin}\, y + dy\, \text{coſ}\, y$. Quia vero eſt

$\text{ſin}\, y = x$, erit coſinus ipſius y ſeu $\text{coſ}\, y = \surd(1 - xx)$; quibus valoribus ſubſtitutis, erit $dx = dy\surd(1 - xx)$, ex qua obtinebitur $dy = \frac{dx}{\surd(1-xx)}$.

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Arcus ergo, cuius ſinus proponitur, differentiale aequatur differentiali ſinus per coſinum diviſo.

196. Cum igitur, ſi p fuerit functio quaecunque ipſius x, atque y denotet arcum, cuius ſinus eſt $= p$, ſeu $y = \text{A}\ \text{ſin}\ p$, ſit huius arcus differentiale $dy = \frac{dp}{\surd(1-pp)}$, ubi $\surd(1-pp)$ exprimit coſinum eiuſdem arcus, inveniri quoque poterit differentiale arcus, cuius coſinus proponitur. Sit enim $y = \text{A}\ \text{coſ}\ x$, erit eiuſdem arcus ſinus $= \surd(1-xx)$, ideoque $y = \text{A}\ \text{ſin}\ \surd(1-xx)$. Facto ergo $p = \surd(1-xx)$, erit $dp = \frac{-xdx}{\surd(1-xx)}$, & $\surd(1-pp) = x$; unde fiet $dy = \frac{-dx}{\surd(1-xx)}$. *Arcus ergo, cuius coſinus proponitur, differentiale aequatur differentiali coſinus negative ſumto, atque per ſinum eiuſdem arcus diviſo.* Quod etiam hoc modo oſtendi poteſt: ſi ſit $y = \text{A}\ \text{coſ}\ x$, ponatur $z = \text{A}\ \text{ſin}\ x$, erit $dz = \frac{dx}{\surd(1-xx)}$ at arcus $y + z$ ſimul ſumti dant arcum conſtantem $90°$, eritque $y + z =$ conſtans ideoque $dy + dz = 0$, ſeu $dy = -dz$; unde fit $dy = \frac{-dx}{\surd(1-xx)}$, ut ante.

197. Si arcus proponatur differentiandus, cuius tangens detur, ita ut ſit $y = \text{A}\ \text{tang}\ x$. Arcus autem cuius tangens eſt x, ſinus erit $= \frac{x}{\surd(1+xx)}$, & coſinus $= \frac{1}{\surd(1+xx)}$. Poſito ergo $\frac{x}{\surd(1+xx)} = p$, ut ſit $\surd(1-pp) = \frac{1}{\surd(1+xx)}$, fiet $y = \text{A}\ \text{ſin}\ p$: unde per regulam modo datam erit $dy = \frac{dp}{\surd(1-pp)}$. At, ob $p = \frac{x}{\surd(1+xx)}$, erit $dp = \frac{dx}{(1+xx)^{\frac{3}{2}}}$;

qui-

quibus valoribus ſubſtitutis fiet $dy = \frac{dx}{1+xx}$. *Arcus ergo, cuius tangens proponitur, differentiale aequatur differentiali tangentis per quadratum ſecantis diviſo.* Eſt enim $\surd(1+xx)$ ſecans, ſi x ſit tangens.

198. Simili modo ſi proponatur arcus, cuius cotangens datur, ita ut ſit $y = \text{A} \cot x$; quia eiuſdem arcus tangens eſt $= \frac{1}{x}$, poſito $\frac{1}{x} = p$, erit $y = \text{A} \operatorname{tang} p$, ac propterea $dy = \frac{dp}{1+pp}$. Cum nunc ſit $dp = \frac{-dx}{xx}$, facta ſubſtitutione, erit $dy = \frac{-dx}{1+xx}$, quod eſt differentiale cotangentis negative ſumtum, atque per quadratum coſecantis diviſum. Porro ſi proponatur $y = \text{A} \sec x$, quia eſt $y = \text{A} \cos \frac{1}{x}$, fiet $dy = \frac{dx}{xx\surd\left(1 - \frac{1}{xx}\right)} = \frac{dx}{x\surd(xx-1)}$. Atque, ſi ſit $y = \text{A} \operatorname{coſec.} x$, erit $y = \text{A} \sin \frac{1}{x}$, ideoque $dy = \frac{-dx}{x\surd(xx-1)}$. Saepe etiam ſinus verſus occurrit, ita ſi proponatur $y = \text{A} \operatorname{ſv.} x$, quia eſt $y = \text{A} \cos(1-x)$, huiuſque arcus ſinus eſt $= \surd(2x - xx)$, fiet $dy = \frac{dx}{\surd(2x-xx)}$.

199. Quamquam ergo arcus, cuius ſinus, vel coſinus, vel tangens, vel cotangens, vel ſecans, vel coſecans, vel denique ſinus verſus datur, eſt quantitas tranſcendens, tamen eius differentiale, ſi per dx dividatur, erit quantitas algebraica, ac propterea quoque eius differentialia ſecunda, tertia, quarta &c. ſi per poteſtates ipſius dx convenientes dividantur.

tur. Ceterum, quo haec differentiatio melius percipiatur, adiunximus ſequentia exempla.

I. Si ſit $y = \mathrm{A}\,\mathrm{ſin}\, 2x\surd(1-xx)$, ponatur $p = 2x\surd(1-xx)$, ut ſit $y = \mathrm{A}\,\mathrm{ſin}\, p$, eritque $dy = \frac{dp}{\surd(1-pp)}$. At eſt $dp = 2\,dx\surd(1-xx) - \frac{2xxdx}{\surd(1-xx)} = \frac{2\,dx(1-2xx)}{\surd(1-xx)}$, & $\surd(1-pp) = 1-2xx$, quibus valoribus ſubſtitutis, erit $dy = \frac{2dx}{\surd(1-xx)}$. Quod etiam inde patet, quod $2x\surd(1-xx)$ ſit ſinus arcus dupli, dum x eſt ſinus ſimpli, erit ergo $y = 2\,\mathrm{A}\,\mathrm{ſin}\, x$, ideoque $dy = \frac{2dx}{\surd(1-xx)}$.

II. Si ſit $y = \mathrm{A}\,\mathrm{ſin}\, \frac{1-xx}{1+xx}$; ponatur $\frac{1-xx}{1+xx} = p$, erit $dp = \frac{-4xdx}{(1+xx)^2}$ & $\surd(1-pp) = \frac{2x}{1+xx}$. Quare cum ſit $dy = \frac{dp}{\surd(1-pp)}$, erit $dy = \frac{-2dx}{1+xx}$.

III. Si ſit $y = \mathrm{A}\,\mathrm{ſin}\surd\frac{1-x}{2}$, ponatur $\surd\frac{1-x}{2} = p$, erit $\surd(1-pp) = \surd\frac{1+x}{2}$, & $dp = \frac{-dx}{4\surd\left(\frac{1-x}{2}\right)}$, unde fit $dy = \frac{dp}{\surd(1-pp)} = \frac{-dx}{2\surd(1-xx)}$.

IV. Si ſit $y = \mathrm{A}\,\mathrm{tang}\, \frac{2x}{1-xx}$, facto $p = \frac{2x}{1-xx}$, erit $1+pp = \frac{(1+xx)^2}{(1-xx)^2}$, & $dp = \frac{2dx(1+xx)}{(1-xx)^2}$. Quare

 cum

cum ſit $dy = \frac{dp}{1+pp}$ per regulam tangentium (197); erit $dy = \frac{2dx}{1+xx}$.

V. Si ſit $y = \text{A tang} \frac{\sqrt{(1+xx)}-1}{x}$, poſito $p = \frac{\sqrt{(1+xx)}-1}{x}$, fiet $pp = \frac{2+xx-2\sqrt{(1+xx)}}{xx}$, & $1+pp = \frac{2+2xx-2\sqrt{(1+xx)}}{xx} = \frac{2(\sqrt{(1+xx)}-1)\sqrt{(1+xx)}}{xx}$. Atqui $dp = \frac{-dx}{xx\sqrt{(1+xx)}} + \frac{dx}{xx} = \frac{dx(\sqrt{(1+xx)}-1)}{xx\sqrt{(1+xx)}}$. Quare cum ſit $dy = \frac{dp}{1+pp}$, fiet $dy = \frac{dx}{2(1+xx)}$; quod etiam inde intelligitur, quod ſit $\text{A tang} \frac{\sqrt{(1+xx)}-1}{x} = \frac{1}{2}\text{A tang}\, x$.

VI. Si ſit $y = e^{\text{A ſin}\, x}$, haec formula quoque per praecedentia differentiabitur: fiet enim $dy = e^{\text{A ſin}\, x} \frac{dx}{\sqrt{(1-xx)}}$. Hoc ergo modo omnes functiones ipſius x, in quas praeter logarithmos atque exponentiales quantitates etiam arcus circulares ingrediuntur, differentiari poterunt.

200. Quoniam differentialia arcuum per dx diviſa ſunt quantitates algebraicae, eorum differentialia ſecunda & ſequentia per ea, quae de functionum algebraicarum differentiatione expoſuimus, invenientur. Sit $y = \text{A ſin}\, x$, quia eſt $dy = \frac{dx}{\sqrt{(1-xx)}}$, erit $\frac{dy}{dx} = \frac{1}{\sqrt{(1-xx)}}$, cuius differentiale dabit valorem pro

ddy

$\frac{ddy}{dx^2}$, ſi quidem dx ſumatur conſtans: unde differentialia ipſius y cuiuſvis ordinis ita ſe habebunt.

Si ſit $y = \text{A} \sin x$; erit

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1-xx)}}; \text{ \& ſumto } dx \text{ conſtante}$$

$$\frac{ddy}{dx^2} = \frac{x}{(1-xx)^{\frac{3}{2}}}$$

$$\frac{d^3y}{dx^3} = \frac{1+2xx}{(1-xx)^{\frac{5}{2}}}$$

$$\frac{d^4y}{dx^4} = \frac{9x+6x^3}{(1-xx)^{\frac{7}{2}}}$$

$$\frac{d^5y}{dx^5} = \frac{9+72x^2+24x^4}{(1-xx)^{\frac{9}{2}}}$$

$$\frac{d^6y}{dx^6} = \frac{225x+600x^3+120x^5}{(1-xx)^{\frac{11}{2}}}$$

&c.

unde concludimus ut ſupra §. 177. fore generaliter:

$$\frac{d^{n+1}y}{dx^{n+1}} = \frac{1.2.3.\ldots\ldots n}{(1-xx)^{n+\frac{1}{2}}} \text{ in}$$

$$\left(x^n + \frac{1}{2}\cdot\frac{n(n-1)}{1.\ 2}x^{n-2} + \frac{1.3}{2.4}\cdot\frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}x^{n-4} + \frac{1.3.5}{2.4.6}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1.\ 2.\ 3.\ 4.\ 5.\ 6}x^{n-6} + \text{\&c.}\right)$$

201. Superſunt quantitates, quae ex harum inverſione naſcuntur, ſcilicet ſinus, tangenteſve arcuum datorum, quas quomodo differentiare oporteat, oſtendamus. Sit igitur x arcus circuli,

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& ſin x denotet eius ſinum, cuius differentiale inveſtigemus. Ponamus $y = \text{ſin}\, x$, ac poſito $x + dx$ loco x, quia y abit in $y + dy$, erit $y + dy = \text{ſin}(x + dx)$, & $dy = \text{ſin}(x + dx) - \text{ſin}\, x$. Eſt autem $\text{ſin}(x + dx) = \text{ſin}\, x . \text{coſ}\, dx + \text{coſ}\, x . \text{ſin}\, dx$. atque cum ſit, uti in introductione oſtendimus

$$\text{ſin}\, z = \frac{z}{1} - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \&c.$$

$$\text{coſ}\, z = 1 - \frac{z^2}{1.2.} + \frac{z^4}{1.2.3.4} - \&c.$$

erit reiectis terminis evaneſcentibus $\text{coſ}\, dx = 1$ & $\text{ſin}\, dx = dx$, unde fit $\text{ſin}(x + dx) = \text{ſin}\, x + dx\, \text{coſ}\, x$. Quare, poſito $y = \text{ſin}\, x$, erit $dy = dx\, \text{coſ}\, x$. *Differentiale ergo ſinus arcus cuiuſvis aequatur differentiali arcus per coſinum multiplicato.* Si igitur fuerit p functio quaecunque ipſius x, erit ſimili modo $d . \text{ſin}\, p = dp\, \text{coſ}\, p$.

202. Similiter ſi proponatur $\text{coſ}\, x$, ſeu coſinus arcus x, cuius differentiale inveſtigari oporteat. Ponatur $y = \text{coſ}\, x$, & poſito $x + dx$ loco x, fiet $y + dy = \text{coſ}(x + dx)$. Eſt vero $\text{coſ}(x + dx) = \text{coſ}\, x . \text{coſ}\, dx - \text{ſin}\, x . \text{ſin}\, dx$, & quia ut modo vidimus eſt $\text{coſ}\, dx = 1$ & $\text{ſin}\, dx = dx$, erit $y + dy = \text{coſ}\, x - dx\, \text{ſin}\, x$, ideoque $dy = -dx\, \text{ſin}\, x$. *Quare differentiale coſinus cuiuſque arcus aequatur differentiali arcus negative ſumto per ſinum eiuſdem arcus multiplicato.* Sic ſi p fuerit functio quaecunque ipſius x, erit $d . \text{coſ}\, p = -dp\, \text{ſin}\, p$. Hae differentiationes quoque ex antecedentibus elici poſſunt hoc modo: ſi fuerit $y = \text{ſin}\, p$, erit $p = \text{A}\, \text{ſin}\, y$; & $dp = \frac{dy}{\surd(1 - yy)}$; at ob $y = \text{ſin}\, p$, erit $\text{coſ}\, p = \surd(1 - yy)$, quo valore ſubſtituto erit $dp = \frac{dy}{\text{coſ}\, p}$ & $dy = dp\, \text{coſ}\, p$, ut ante. Pari modo ſi ſit $y = \text{coſ}\, p$, erit $\surd(1 - yy) = \text{ſin}\, p$, & $p = \text{A}\, \text{coſ}\, y$, ideoque $dp = \frac{-dy}{\surd(1 - yy)} = \frac{-dy}{\text{ſin}\, p}$, unde fit ut ante $dy = -dp\, \text{ſin}\, p$.

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203. Si fuerit $y = \text{tang}\, x$, erit $dy = \text{tang}\,(x + dx) - \text{tang}\, x$; at eſt $\text{tang}\,(x + dx) = \frac{\text{tang}\, x + \text{tang}\, dx}{1 - \text{tang}\, x . \text{tang}\, dx}$, a qua fractione ſi tangens x ſubtrahatur, remanebit $dy = \frac{\text{tang}\, dx\,(1 + \text{tang}\, x . \text{tang}\, x)}{1 - \text{tang}\, x . \text{tang}\, dx}$. Verum arcus evaneſcentis dx tangens ipſi arcui eſt aequalis, ideoque $\text{tang}\, dx = dx$, & denominator $1 - dx\,\text{tang}\, x$, abit in unitatem: quocirca fiet $dy = dx\,(1 + \text{tang}\, x^2)$. Eſt vero $1 + \text{tang}\, x^2 = \text{ſec}\, x^2 = \frac{1}{\text{coſ}\, x^2}$, denotante $\text{coſ}\, x^2$ quadratum coſinus ipſius x: conſequenter ſi fuerit $y = \text{tang}\, x$, erit $dy = dx\,\text{ſec}\, x^2 = \frac{dx}{\text{coſ}\, x^2}$. Quod differentiale quoque per differentiationem ſinuum & coſinuum inveniri poteſt; cum enim ſit $\text{tang}\, x = \frac{\text{ſin}\, x}{\text{coſ}\, x}$, erit

$$dy = \frac{dx\,\text{coſ}\, x . \text{coſ}\, x + dx\,\text{ſin}\, x . \text{ſin}\, x}{\text{coſ}\, x^2} = \frac{dx}{\text{coſ}\, x^2}.$$

ob $\text{ſin}\, x^2 + \text{coſ}\, x^2 = 1$.

204. Aliter etiam hoc differentiale invenitur. Cum ſit $y = \text{tang}\, x$, erit $x = \text{A}\,\text{tang}\, y$, & per praecepta ſuperiora fiet $dx = \frac{dy}{1 + yy}$. At cum ſit $y = \text{tang}\, x$, erit $\surd(1 + yy) = \text{ſec}\, x = \frac{1}{\text{coſ}\, x}$, ideoque $dx = dy\,\text{coſ}\, x^2$, & $dy = \frac{dx}{\text{coſ}\, x^2}$, ut ante. *Tangentis ergo cuiuſvis arcus differentiale aequatur differentiali arcus diviſo per quadratum coſinus eiuſdem arcus.* Simili modo ſi proponatur $y = \text{cot}\, x$, fiet $x = \text{A}\,\text{cot}\, y$, & $dx = \frac{-dy}{1 + yy}$. At vero erit

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$\surd(1+yy) = \text{coſec}\, x = \frac{1}{\text{ſin}\, x}$, unde habebitur $dx = -dy\, \text{ſin}\, x^2$, & $dy = \frac{-dx}{\text{ſin}\, x^2}$. *Cotangentis ergo cuiuſvis arcus differentiale aequatur differentiali arcus negative ſumto ac per quadratum ſinus eiuſdem arcus diviſo*. Vel quia eſt $\cot x = \frac{\text{coſ}\, x}{\text{ſin}\, x}$, fiet hanc fractionem differentiando:

$$dy = \frac{-dx\, \text{ſin}\, x^2 - dx\, \text{coſ}\, x^2}{\text{ſin}\, x^2} = \frac{-dx}{\text{ſin}\, x^2},$$

uti modo invenimus.

205. Si proponatur ſecans arcus, ut ſit $y = \text{ſec}\, x$, quia erit $y = \frac{1}{\text{coſ}\, x}$, erit $dy = \frac{dx\, \text{ſin}\, x}{\text{coſ}\, x^2} = dx\, \text{tang}\, x.\, \text{ſec}\, x$. Simili modo ſi fuerit $y = \text{coſec}\, x$, ob $y = \frac{1}{\text{ſin}\, x}$, erit $dy = \frac{-dx\, \text{coſ}\, x}{\text{ſin}\, x^2} = -dx \cot x\, \text{coſec}\, x$, pro quibus caſibus peculiares regulas formare ſuperfluum foret. Si ſinus verſus arcus proponatur $y = \text{ſv}.x$, quia eſt $y = 1 - \text{coſ}\, x$, erit $dy = dx\, \text{ſin}\, x$. Omnes ergo caſus, quibus linea quaepiam recta ad arcum relata proponitur, quia ſemper per ſinum coſinumve exprimi poteſt, ſine difficultate differentiari poterunt. Neque vero tantum differentialia prima, ſed etiam ſecunda & ſequentia per regulas datas invenientur. Ponamus eſſe $y = \text{ſin}\, x$ & $z = \text{coſ}\, x$, atque dx eſſe conſtans: erit ut ſequitur:

$y = \text{ſin}\, x$	$z = \text{coſ}\, x$
$dy = dx\, \text{coſ}\, x$	$dz = -dx\, \text{ſin}\, x$
$ddy = -dx^2\, \text{ſin}\, x$	$ddz = -dx^2\, \text{coſ}\, x$
$d^3y = -dx^3\, \text{coſ}\, x$	$d^3z = dx^3\, \text{ſin}\, x$
$d^4y = dx^4\, \text{ſin}\, x$	$d^4z = dx^4\, \text{coſ}\, x$
&c.	&c.

206.

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206. Simili modo inveniri poterunt differentialia omnium ordinum tangentis arcus x. Sit enim $y = \tang x = \frac{\ſin x}{\coſ x}$, & ponatur dx conſtans, erit

$$y = \frac{\ſin x}{\coſ x}$$

$$\frac{dy}{dx} = \frac{1}{\coſ x^2}$$

$$\frac{ddy}{dx^2} = \frac{2 \ſin x}{\coſ x^3}$$

$$\frac{d^3y}{dx^3} = \frac{6}{\coſ x^4} - \frac{4}{\coſ x^2}$$

$$\frac{d^4y}{dx^4} = \frac{24 \ſin x}{\coſ x^5} - \frac{8 \ſin x}{\coſ x^3}$$

$$\frac{d^5y}{dx^5} = \frac{120}{\coſ x^6} - \frac{120}{\coſ x^4} + \frac{16}{\coſ x^2}$$

$$\frac{d^6y}{dx^6} = \frac{720 \ſin x}{\coſ x^7} - \frac{480 \ſin x}{\coſ x^5} + \frac{32 \ſin x}{\coſ x^3}$$

$$\frac{d^7y}{dx^7} = \frac{5040}{\coſ x^8} - \frac{6720}{\coſ x^6} + \frac{2016}{\coſ x^4} - \frac{64}{\coſ x^2}$$

&c.

207. Functiones ergo quaecunque, in quas ſinus vel coſinus arcuum ingrediuntur, per haec praecepta differentiari poterunt, uti ex ſequentibus exemplis videre licet.

I. Si ſit $y = 2 \ſin x . \coſ x = \ſin 2x$

Erit $dy = 2\, dx \coſ x^2 - 2\, dx \ſin x^2 = 2\, dx \coſ 2x$.

II. Si ſit $y = \sqrt{\frac{1 - \coſ x}{2}}$, vel $y = \ſin \frac{1}{2} x$

Erit

Erit $dy = \frac{dx \sin x}{2\sqrt{2(1-\cos x)}}$. Cum autem ſit $\sqrt{2(1-\cos x)} = 2\sin\frac{1}{2}x$, & $\sin x = 2\sin\frac{1}{2}x.\cos\frac{1}{2}x$; fiet $dy = \frac{1}{2}dx.\cos\frac{1}{2}x$, uti ex forma $y = \sin\frac{1}{2}x$ immediate ſequitur.

III. Si ſit $y = \cos l\frac{1}{x}$; erit, poſito $l\frac{1}{x} = p$, $y = \cos p$, & $dy = -dp\sin p$. At, ob $p = l1 - lx$, erit $dp = \frac{-dx}{x}$; ideoque $dy = \frac{dx}{x}\sin l\frac{1}{x}$

IV. Si ſit $y = e^{\sin x}$; erit $dy = e^{\sin x}dx\cos x$.

V. Si ſit $y = e^{\frac{-n}{\cos x}}$; erit $dy = -\frac{e^{\frac{-n}{\cos x}}ndx\sin x}{\cos x^2}$.

VI. Si ſit $y = l\left(1-\sqrt{\left(1-e^{\frac{-n}{\sin x}}\right)}\right)$; ponatur $e^{\frac{-n}{\sin x}} = p$; atque ob $y = l(1-\sqrt{(1-p)})$, erit

$$dy = \frac{dp}{2(1-\sqrt{(1-p)})\sqrt{(1-p)}}. \text{ At eſt } dp = \frac{e^{\frac{-n}{\sin x}}ndx\cos x}{\sin x^2}.$$

Quo valore ſubſtituto prodibit

$$dy = \frac{-ne^{\frac{-n}{\sin x}}dx\cos x}{2\sin x^2\left(1-\sqrt{\left(1-e^{\frac{-n}{\sin x}}\right)}\right)\sqrt{\left(1-e^{\frac{-n}{\sin x}}\right)}}$$

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CAPUT VII.

DE DIFFERENTIATIONE FUNCTIONUM DUAS PLURESVE VARIABILES INVOLVENTIUM.

208.

Si duae pluresve quantitates variabiles x, y, z a ſe invicem prorſus non pendeant, fieri poteſt, ut etiamſi omnes ſint variabiles, tamen dum una creſcit decreſcitve, reliquae maneant invariatae: quia enim nullum nexum inter ſe habere ponuntur, immutatio unius reliquas non afficit. Neque ergo differentialia quantitatum y & z pendebunt a differentiali ipſius x; ideoque dum x differentiali ſuo dx augetur, quantitates y & z, vel eaedem manere, vel quomodocunque pro lubitu variari poſſunt. Quodſi igitur differentiale quantitatis x ſtatuatur dx, reliquarum quantitatum differentialia dy & dz manent indeterminata, atque pro arbitrio noſtro vel prorſus nihil, vel infinite parva ad dx quamvis rationem tenentia denotabunt.

209. Plerumque autem litterae y & z functiones ipſius x vel incognitas, vel quarum relatio ad x non ſpectatur, ſignificare ſolent, hocque caſu earum differentialia dy & dz certam ad dx relationem habebunt. Sive autem y & z pendeant ab x ſive ſecus, ratio differentiationis, quam hic ſpectamus, eodem redit. Quaerimus enim functionis, quae ex pluribus variabilibus $x, y, \& z$ utcunque ſit formata, differentiale, quod accipit, dum ſingulae variabiles $x, y, \& z$ ſuis differentialibus $dx, dy, \& dz$ creſcunt. Ad hoc ergo inveniendum in functione propoſita ubique loco variabilium quantitatum x, y, z ſcribatur reſpective $x + dx$; $y + dy$; $z + dz$, & ab expreſſione hoc modo reſultante auferatur ipſa functio propoſita: reſiduum dabit ipſum differentiale, quod quaeritur,

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quemadmodum ex natura differentialium luculenter conſtat.

210. Sit X functio ipſius x, eiuſque differentiale, ſeu augmentum, dum x differentiali ſuo dx creſcit, ſit $= Pdx$. Deinde ſit Y functio ipſius y, eiuſque differentiale $= Qdy$, quod augmentum Y accipit, dum y abit in $y + dy$: atque Z ſit functio ipſius z, eiuſque differentiale ſit $= Rdz$, quae differentialia Pdx, Qdy, Rdz ex natura functionum X, Y, & Z ope praeceptorum ſupra datorum inveniri poterunt. Quod ſi ergo propoſita fuerit haec quantitas $X + Y + Z$, quae utique erit functio trium variabilium x, y, & z, eius differentiale erit $= Pdx + Qdy + Rdz$. Utrum autem haec tria differentialia ſint inter ſe homogenea nec ne perinde eſt. Termini enim qui continent poteſtates ipſius dx prae Pdx aeque evaneſcunt, ac ſi reliqua membra Qdy & Rdz abeſſent, ſimiliſque eſt ratio terminorum, qui in differentiatione functionum Y & Z ſunt neglecti.

211. Retineant X, Y & Z eaſdem ſignificationes, ſitque propoſita iſta functio XYZ ipſarum x, y & z, cuius differentiale inveſtigari oporteat. Quoniam, ſi $x + dx$ loco x, $y + dy$ loco y, & $z + dz$ loco z ſcribatur, abit X in $X + Pdx$; Y in $Y + Qdy$; & Z in $Z + Rdz$, ipſa functio propoſita XYZ abibit in

$$(X + Pdx)(Y + Qdy)(Z + Rdz)$$
$$= XYZ + YZP\,dx + XZQ\,dy + XYR\,dz$$
$$+ ZPQdxdy + YPRdxdz + XQRdydz + PQRdxdydz.$$

At quia dx, dy, & dz ſunt infinite parva, ſive inter ſe ſint homogenea ſive non; ultimus terminus prae unoquoque praecedentium evaneſcit. Deinde terminus $ZPQdxdy$ tam prae $YZPdx$ quam prae $XZQdy$ evaneſcit; atque ob eandem rationem termini $YPRdxdz$ & $XQZdydz$ evaneſcent. Ablata ergo ipſa functione propoſita XYZ, erit eius differentiale

$$= YZPdx + XZQdy + XYRdz.$$

212. Exempla haec functionum trium variabilium x, y, & z, quibus pro lubitu quiſque plura adiicere poteſt, ſufficiunt

ciunt ad oſtendendum, ſi functio quaecunque trium variabilium x, y, & z proponatur, utcunque etiam hae variabiles inter ſe fuerint permixtae, eius differentiale ſemper huiuſmodi formam eſſe habiturum $pdx + qdy + rdz$: ubi p, q, & r futurae ſint ſingulae functiones, vel omnium trium variabilium x, y, & z, vel binarum, vel unius tantum, prout ratio compoſitionis, qua functio propoſita ex variabilibus x, y, & z atque conſtantibus formatur, fuerit comparata. Simili modo, ſi proponatur functio quatuor pluriumve variabilium x, y, z, & v, eius differentiale ſemper huius modi formam habebit

$$pdx + qdy + rdz + sdv.$$

213. Conſideremus primum functionem duarum tantum variabilium x & y, quae ſit $= V$, cuius ergo differentiale ita ſe habebit, ut ſit $dV = pdx + qdy$. Si igitur quantitas y aſſumeretur conſtans, foret $dy = 0$, ideoque functionis V differentiale eſſet pdx: ſin autem x ſtatueretur conſtans, ut eſſet $dx = 0$, ſolaque y maneret variabilis, tum ipſius V differentiale prodiret $= qdy$. Cum igitur utraque quantitate x & y variabili poſita ſit $dV = pdx + qdy$, iſta regula pro differentianda functione V duas variabiles x & y involvente reſultabit: *Ponatur primum ſola* x *variabilis, altera vero* y *tanquam conſtans tractetur, & quaeratur ipſius* V *differentiale, quod ſit* $=$ pdx. *Deinde ponatur ſola quantitas* y *variabilis, altera* x *pro conſtanti habita, & quaeratur ipſius* V *differentiale, quod ſit* $=$ qdy. *Quibus factis, poſita utraque quantitate* x *&* y *variabili, fiet* dV $=$ pdx $+$ qdy.

214. Simili modo, cum functionis trium variabilium x, y, & z, quae ſit $= V$, differentiale huiuſmodi habeat formam $dV = pdx + qdy + rdz$, manifeſtum eſt, ſi ſola quantitas x fuiſſet variabilis poſita, reliquae vero y & z conſtantes manſiſſent, ob $dy = 0$ & $dz = 0$, prodiiſſet ipſius V differentiale $= pdx$. Pari modo inveniretur differentiale ipſius $V = qdy$, ſi x & z eſſent conſtantes ſolaque y poneretur variabilis; atque ſi x & y tanquam conſtantes tractarentur ſola-

que z ſtatueretur variabilis, prodiret differentiale ipſius $V = rdz$. Quare ad functionem trium pluriumve variabilium differentiandam, conſideretur ſeorſim quaelibet quantitas variabilis, & functio pro qualibet differentietur, quaſi reliquae omnes eſſent conſtantes; tum ſingula haec differentialia, quae ex ſingulis quantitatibus variabilibus ſunt inventa, colligantur, eritque aggregatum differentiale quaeſitum functionis propoſitae.

215. In hac regula, quam pro differentiatione functionis quotcunque variabilium invenimus, continetur demonſtratio regulae ſupra §. 170 datae generalis, cuius ope functio quaecunque unicam variabilem complectens differentiari poteſt. Si enim pro ſingulis partibus ibi commemoratis totidem litterae diverſae collocentur, functio ſpeciem induet functionis totidem diverſarum variabilium, atque adeo modo hic praeſcripto differentiabitur, ſucceſſive unamquamque partem, quaſi ſola eſſet variabilis, tractando, cunctaque differentialia, quae ex ſingulis partibus oriuntur, in unam ſummam coniiciendo: quae ſumma erit differentiale quaeſitum, poſtquam pro ſingulis litteris valores fuerint reſtituti. Haec ergo regula latiſſime patet, atque etiam ad functiones plurium variabilium, quomodocunque fuerint comparatae, extenditur. Unde eius uſus per univerſum calculum differentialem eſt ampliſſimus.

216. Inventa ergo regula generali, cuius ope functiones quotcunque variabilium differentiari poſſunt, eius uſum in nonnullis exemplis oſtendiſſe iuvabit.

I. Si fuerit $V = xy$; erit $dV = xdy + ydx$.

II. Si fuerit $V = \frac{x}{y}$; erit $dV = \frac{dx}{y} - \frac{xdy}{yy}$.

III. Si fuerit $V = \frac{y}{\sqrt{(aa - xx)}}$; erit

$$dV = \frac{dy}{\sqrt{(aa - xx)}} + \frac{yxdx}{(aa - xx)^{\frac{3}{2}}}.$$

IV.

IV. Si fuerit $V=(\alpha x+\beta y+\gamma)^m(\delta x+\varepsilon y+\zeta)^n$; erit

$$dV = m(\alpha x+\beta y+\gamma)^{m-1}(\delta x+\varepsilon y+\zeta)^n(\alpha dx+\beta dy)$$
$$+ n(\alpha x+\beta y+\gamma)^m(\delta x+\varepsilon y+\zeta)^{n-1}(\delta dx+\varepsilon dy),$$

ſive $dV=(\alpha x+\beta y+\gamma)^{m-1}(\delta+\varepsilon y+\zeta)^{n-1}$ in

$$\left(\begin{matrix}m\alpha\delta\\ n\alpha\delta\end{matrix}\, xdx \begin{matrix}+m\beta\delta\\ +n\alpha\varepsilon\end{matrix}\, xdy \begin{matrix}+m\alpha\varepsilon\\ +n\beta\delta\end{matrix}\, ydx\right.$$
$$\left.\begin{matrix}+m\beta\varepsilon\\ +n\beta\varepsilon\end{matrix}\, ydy \begin{matrix}+m\alpha\zeta\\ +n\gamma\delta\end{matrix}\, dx \begin{matrix}+m\beta\zeta\\ +n\gamma\varepsilon\end{matrix}\, dy\right).$$

V. Si fuerit $V=ylx$; erit $dV=dylx+\frac{ydx}{x}$.

VI. Si fuerit $V=x^y$; erit $dV=yx^{y-1}dx+x^y dylx$.

VII. Si fuerit $V=A\tang\frac{y}{x}$; erit $dV=\frac{xdy-ydx}{xx+yy}$.

VIII. Si fuerit $V=\text{ſin}x.\text{coſ}y$; erit $dV=dx\,\text{coſ}x.\text{coſ}y-dy\,\text{ſin}x.\text{ſin}y$.

IX. Si fuerit $V=\frac{e^z y}{\sqrt{(xx+yy)}}$; erit

$$dV=\frac{e^z ydz}{\sqrt{(xx+yy)}}+\frac{e^z(xxdy-yxdx)}{(xx+yy)\sqrt{(xx+yy)}}.$$

X. Si fuerit $V=e^z A\,\text{ſin}\,\frac{x-\sqrt{(xx-yy)}}{x+\sqrt{(xx-yy)}}$,

reperietur $dV=e^z dz A\,\text{ſin}\,\frac{x-\sqrt{(xx-yy)}}{x+\sqrt{(xx-yy)}}$

$$+e^z.\frac{xydy-yydx}{(x+\sqrt{(xx-yy)})(xx-yy)^{\frac{1}{4}}\sqrt{x}}$$

217. Quoniam vidimus, ſi V fuerit functio quaecunque binarum variabilium x & y, eius differentiale huiuſmodi habiturum eſſe formam $dV=Pdx+Qdy$, in qua ſint P & Q functiones a fuuctione V pendentes per eamque determinatae: ſequitur has duas quantitates P & Q certo quodam modo

do etiam a ſe invicem pendere, propterea quod utraque ab eadem functione V pendet. Quicunque igitur ſit iſte nexus inter quantitates finitas P & Q, quem deinceps inveſtigabimus, perſpicuum eſt, non omnes formulas differentiales huiuſmodi $Pdx + Qdy$, in quibus P & Q pro lubitu ſint ex x & y formatae, poſſe eſſe differentialia cuiuſpiam functionis finitae V ipſarum x & y. Niſi enim ea relatio inter functiones P & Q intercedat, quam natura differentiationis requirit, huiuſmodi differentiale $Pdx + Qdy$ oriri plane per differentiationem non potuit, ideoque viciſſim integrale non habebit.

218. In integratione igitur plurimum intereſt noſſe hanc relationem inter quantitates P & Q, ut differentialia, quae revera ex differentiatione functionis cuiuſpiam finitae ſunt orta, dignoſci queant ab iis, quae ad libitum ſunt formata, atque nulla integralia admittunt. Quanquam autem hic nondum integrationis negotium ſuſcipimus, tamen ad naturam differentialium realium penitius inſpiciendam conveniet hanc relationem inveſtigari; quippe cuius cognitio non ſolum ad calculum integralem, ad quem hic viam paramus, eſt maxime neceſſaria, ſed etiam in ipſo calculo differentiali inſignem lucem accendit. Primum igitur patet, ſi V ſit functio duarum variabilium x & y, in eius differentiali $Pdx + Qdy$ utriuſque differentiale dx & dy ineſſe oportere. Neque ergo poteſt eſſe $P = 0$ neque $Q = 0$. Hinc ſi P fuerit functio ipſarum x & y, formula Pdx nullius quantitatis finitae poterit eſſe differentiale, ſeu nulla extat quantitas finita, cuius differentiale ſit Pdx.

219. Sic nulla datur quantitas finita V ſive algebraica ſive tranſcendens, cuius differentiale ſit $yxdx$, ſi quidem ſit y quantitas variabilis ab x non pendens. Si enim ponamus dari eiuſmodi quantitatem finitam V, quia y in eius differentiale ingreditur, neceſſe eſt, ut y quoque in ipſa quantitate V inſit; verum ſi V contineret y, ob variabilitatem ipſius y ne-

neceſſario quoque in differentiali ipſius V differentiale dy ineſſe deberet. Quod tamen cum non adſit, fieri nequit, ut differentiale $yxdx$ ex cuiuſpiam quantitatis finitae differentiatione ſit ortum. Cum igitur pateat formulam $Pdx + Qdy$, ſi Q ſit o, & P contineat y, differentiale reale eſſe non poſſe, ſimul intelligitur, quantitati Q non pro lubitu valorem tribui poſſe, ſed eum a valore ipſius P pendere.

220. Quo igitur hanc relationem inter P & Q in differentiali $dV = Pdx + Qdy$ inveſtigemus, ponamus primo V eſſe functionem nullius dimenſionis ipſarum x & y: a caſibus enim particularibus ad relationem generalem aſcendamus. Quod ſi ergo ponamus $y = tx$, ex functione V quantitas x prorſus evaneſcet, prodibitque functio ipſius t tantum, quae ſit $= T$, cuius differentiale erit $= \Theta dt$, exiſtente Θ functione ipſius t. Ponamus igitur quoque in differentiali $Pdx + Qdy$ ubique $y = tx$, & $dy = tdx + xdt$, quo facto prodibit

$$Pdx + Qtdx + Qxdt;$$

in quo cum dx non contineatur, neceſſe eſt ut ſit $P + Qt = o$; ideoque $Q = -\frac{P}{t} = -\frac{Px}{y}$, ſeu erit $Px + Qy = o$, unde relatio inter P & Q pro hoc caſu innoteſcit. Deinde debet eſſe $\Theta = Qx$, ideoque $Qx =$ functioni ipſius t, hoc eſt functioni nullius dimenſionis ipſarum x & y. Atque ob $Q = \frac{\Theta}{x}$ fiet $P = -\frac{\Theta y}{xx}$, & tam Px quam Qy erunt functiones nullius dimenſionis ipſarum x & y.

221. Si igitur functio nullius dimenſionis ipſarum x & y, quae ſit $= V$, differentietur, eius differentiale $dV = Pdx + Qdy$, ſemper ita erit comparatum, ut ſit $Px + Qy = o$. Hoc eſt ſi in differentiali loco differentialium dx & dy ſcribantur x & y, reſultabit quantitas $= o$: uti in his exemplis uſu venire patet:

I.

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I. Sit $V = \frac{x}{y}$; erit $dV = \frac{ydx - xdy}{yy}$, atque poſito x loco dx & y loco dy, erit $\frac{yx - xy}{yy} = 0$.

II. Sit $V = \frac{x}{\sqrt{(xx - yy)}}$, erit $dV = \frac{-yydx + yxdy}{(xx - yy)^{\frac{3}{2}}}$, unde fit $\frac{-yyx + yyx}{(xx - yy)^{\frac{3}{2}}} = 0$.

III. Sit $V = \frac{y + \sqrt{(xx + yy)}}{-y + \sqrt{(xx + yy)}}$, quae eſt functio nullius dimenſionis ipſarum x & y; erit

$dV = \frac{2xxdy - 2xydx}{(\sqrt{(xx + yy)} - y)^2 \sqrt{(xx + yy)}}$, quae forma poſitis x & y loco dx & dy fit $= 0$.

IV. Sit $V = l\frac{x + y}{x - y}$; erit $dV = \frac{2xdy - 2ydx}{xx - yy}$, atque $\frac{2xy - 2yx}{xx - yy} = 0$.

V. Sit $V = \text{A ſin} \frac{\sqrt{(x - y)}}{\sqrt{(x + y)}}$, erit $dV = \frac{ydx - xdy}{(x + y)\sqrt{2y(x - y)}}$, quae formula eadem proprietate gaudet.

222. Contemplemur nunc alias functiones homogeneas, ſitque V functio n dimenſionum ipſarum x & y. Quare ſi ponatur $y = tx$, induet V huiuſmodi formam Tx^n, exiſtente T functione ipſius t, ſitque

$dT = \Theta dt$, erit $dV = x^n \Theta dt + nTx^{n-1} dx$.

Quodſi ergo ſtatuamus:

$dV = Pdx + Qdy$, ob $dy = tdx + xdt$,

fiet $dV = Pdx + Qtdx + Qxdt$:

quae

quae forma quoniam cum illa congruere debet, erit

$$P + Qt = nTx^{n-1} = \frac{nV}{x}, \quad \text{ob} \quad V = Tx^n.$$

Hancobrem ob $t = \frac{y}{x}$, fiet $Px + Qy = nV$, quae aequatio relationem inter P & Q ita definit, ut ſi altera ſit cognita, altera facile inveniatur. Quia porro eſt $Qx = x^n\,\Theta$, erit Qx, ideoque etiam Qy & Px functio n dimenſionum ipſarum x & y.

223. Si ergo in differentiali cuiuſvis functionis homogeneae ipſarum x & y, loco dx & dy, ponatur x & y, quantitas oriunda aequabitur ipſi functioni, cuius differentiale proponebatur, per numerum dimenſionum multiplicatae.

I. Si ſit $V = \sqrt{(xx + yy)}$; erit $n = 1$, & ob

$$dV = \frac{xdx + ydy}{\sqrt{(xx+yy)}}, \text{ fiet } \frac{xx + yy}{\sqrt{(xx+yy)}} = V = \sqrt{(xx+yy)}.$$

II. Si ſit $V = \frac{y^3 + x^3}{y - x}$; erit $n = 2$, &

$$dV = \frac{2y^3dy - 3yyxdy + 3yxxdx - 2x^3dx + y^3dx - x^3dy}{(y-x)^2},$$

Ponatur x pro dx & y pro dy orietur:

$$\frac{2y^4 - 2y^3x + 2yx^3 - 2x^4}{(y-x)^2} = \frac{2y^3 + 2x^3}{y - x} = 2V.$$

III. Si ſit $V = \frac{1}{(yy+xx)^2}$, erit $n = -4$, atque

$dV = -\frac{4ydy + 4xdx}{(yy+xx)^3}$. Quae formula poſitis x & y loco dx & dy abit in $-\frac{4yy + 4xx}{(yy+xx)^3} = -4V$.

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IV. Si ſit $V = xxl\frac{y+x}{y-x}$; erit $n=2$, atque $dV = 2xdxl\frac{y+x}{y-x} + \frac{2xx(ydx - xdy)}{yy-xx}$, facta autem memorata ſubſtitutione oritur $2xxl\frac{y+x}{y-x} = 2V$.

224. Similis proprietas obſervabitur, ſi V fuerit functio homogenea plurium variabilium, ſit ergo V functio quantitatum x, y, z, quae coniunctim ubique n dimenſiones adimpleant; atque differentiale huiuſmodi habebit formam $Pdx + Qdy + Rdz$. Ponatur iam $y = tx$ & $z = sx$, ut ſit $dy = tdx + xdt$, & $dz = sdx + xds$, atque functio V induet hanc formam Ux^n, exiſtente U functione binarum variabilium t & s; hinc ergo ſi ſtatuatur $dU = pdt + qds$, fiet

$$dV = x^n pdt + x^n qds + nUx^{n-1}dx.$$

Prior autem forma dabit

$$dV = Pdx + Qtdx + Qxdt + Rsdx + Rxds:$$

quae cum illa collata praebet

$$P + Qt + Rs = nUx^{n-1} = \frac{nV}{x},$$

unde obtinetur $Px + Qy + Rz = nV$; quae eadem proprietas ad quotcunque plures variabiles extenditur.

225. Si igitur propoſita fuerit functio homogenea quotcunque variabilium x, y, z, v, &c. eius differentiale perpetuo hanc habebit proprietatem, ut ſi loco differentialium dx, dy, dz, dv, &c. ſcribantur reſpective quantitates finitae x, y, z, v, &c. prodeat ipſa functio propoſita per numerum dimenſionum multiplicata. Haecque regula etiam valet, ſi V fuerit functio homogenea unicae tantum variabilis x: Hoc enim caſu erit V poteſtas ipſius x, puta $V = ax^n$, quae eſt functio homogenea n dimenſionum: nulla ſcilicet alia

da-

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datur functio ipſius x, in qua x ubique n dimenſiones conſtituat praeter poteſtatem x^n. Cum igitur ſit $dV = nax^{n-1}dx$, ponatur x loco dx, atque prodibit nax^n, hoc eſt nV. Iſta ergo functionum homogenearum inſignis proprietas diligenter notari meretur, cum in calculo integrali maximam afferat utilitatem.

226. Quo nunc in genere in relationem inter quantitates P & Q, quae differentiale $Pdx + Qdy$ functionis cuiuſcunque V duarum variabilium x & y conſtituunt, inquiramus, ad ſequentia attendi oportebit. Sit igitur V functio quaecunque ipſarum x & y; atque ponamus V abire in R, ſi loco x ponatur $x+dx$; poſito autem $y+dy$ loco y abeat V in S: quodſi autem ſimul $x+dx$ loco x, & $y+dy$ loco y ſcribatur, mutetur V in V'. Cum itaque R oriatur ex V, poſito $x+dx$ loco x, manifeſtum eſt ſi ulterius in R ponatur $y+dy$ loco y, tum prodire V'; idem enim eſt, ac ſi in V ſtatim poneretur $x+dx$ loco x, & $y+dy$ loco y. Simili modo ſi in S ponatur $x+dx$ loco x, quia S iam orta eſt ex V poſito $y+dy$ loco y, denuo prodibit V'; uti ex hoc ſchematiſmo clarius perſpicitur.

Quantitas	abit in	ſi loco	ponatur.
V	R	x	$x+dx$
V	S	y	$y+dy$
V	V'	x y	$x+dx$ $y+dy$
R	V'	y	$y+dy$
S	V'	x	$x+dx$

227. Si igitur V ita differentietur, ut tantum x tanquam variabilis, y vero tanquam conſtans tractetur, quia po-

 ſito

ſito $x+dx$ loco x, functio V abit in R, eius differentiale erit $=R-V$; at ex forma $dV=Pdx+Qdy$, ſequitur idem differentiale fore $=Pdx$, unde erit $R-V=Pdx$. Quod ſi iam loco y ponatur $y+dy$, x vero tanquam conſtans tractetur, quia R abit in V^I & V in S, quantitas $R-V$ abibit in V^I-S; ideoque ipſius $R-V=Pdx$ differentiale, quod oritur ſi ſola y variabilis aſſumatur, erit $V^I-R-S+V$. ſimili modo, cum poſito $y+dy$ loco y, abeat V in S, erit $S-V$ differentiale ipſius V poſita ſola y variabili, eritque propterea $S-V=Qdy$; nunc quia loco x poſito $x+dx$, tranſit S in V^I & V in R, quantitas $S-V$ abibit in V^I-R; atque ipſius $S-V=Qdy$ differentiale, quod oritur ſi ſola x variabilis ſtatuatur, erit $=V^I-R-S+V$, quod prorſus congruit cum differentiali ante invento.

228. Ex hac convenientia deducitur ſequens concluſio: Si functionis V cuiuſcunque binarum variabilium x & y differentiale fuerit $dV=Pdx+Qdy$, tum differentiale ipſius Pdx quod oritur ſi ſola quantitas y tanquam variabilis, x vero tanquam conſtans tractetur, aequale erit differentiali ipſius Qdy, quod oritur ſi ſola quantitas x tanquam variabilis, y vero tanquam conſtans tractetur. Si ſcilicet poſita ſola y variabili fuerit $dP=Zdy$ erit differentiale ipſius Pdx praeſcripto modo ſumtum $=Zdxdy$; atque poſita ſola x variabili erit quoque $dQ=Zdx$; ſic enim differentiale ipſius Qdy praeſcripto modo ſumtum fiet quoque $=Zdxdy$. Hacque ratione intelligitur relatio, quae inter quantitates P & Q intercedit, atque paucis verbis in hoc conſiſtit, ut differentiale ipſius Pdx poſito x conſtante aequale ſit differentiali ipſius Qdy poſito y conſtante.

229. Iſta inſignis proprietas clarius perſpicietur, ſi eam nonnullis exemplis illuſtremus.

I. Sit igitur $V=yx$; erit $dV=ydx+xdy$, ideoque $P=y$ & $Q=x$; unde poſito x conſtante erit $d.Pdx=dxdy$, & poſito y conſtante erit $d.Qdy=dxdy$, ſicque haec duo differentialia inter ſe aequantur.

II. Sit $V = \sqrt{(xx + 2xy)}$; erit $dV = \frac{xdx + ydx + xdy}{\sqrt{(xx + 2xy)}}$, ideoque $P = \frac{x + y}{\sqrt{(xx + 2xy)}}$, & $Q = \frac{x}{\sqrt{(xx + 2xy)}}$, unde poſito x conſtante erit $d.Pdx = \frac{xydxdy}{(xx + 2xy)^{\frac{3}{2}}}$, & poſito y conſtante erit $d.Qdy = \frac{xydxdy}{(xx + 2xy)^{\frac{3}{2}}}$.

III. Sit $V = x \operatorname{ſin} Ay + y \operatorname{ſin} Ax$; eritque $dV = dx \operatorname{ſin} Ay + xdy \operatorname{coſ} y + dy \operatorname{ſin} Ax + ydx \operatorname{coſ} x$.

Quare erit

$$Pdx = dx \operatorname{ſin} Ay + ydx \operatorname{coſ} x,$$

& $$Qdy = dy \operatorname{ſin} Ax + xdy \operatorname{coſ} y.$$

Poſito ergo x conſtante erit

$$d.Pdx = dxdy \operatorname{coſ} y + dxdy \operatorname{coſ} x,$$

& poſito y conſtante erit

$$d.Qdy = dxdy \operatorname{coſ} y + dxdy \operatorname{coſ} x.$$

IV. Sit $V = x^y$; erit $dV = x^y dylx + yx^{y-1} dx$, atque $Pdx = yx^{y-1}dx$, & $Qdy = x^y dylx$.

Quamobrem poſito x conſtante habebitur

$$d.Pdx = x^{y-1} dxdy + yx^{y-1} dxdylx,$$

& poſito y conſtante erit

$$d.Qdy = yx^{y-1} dxdylx + x^{y-1} dxdy.$$

230. Iſta proprietas etiam hoc modo enunciari poteſt, unde eximia omnium functionum, quae duas variabiles involvunt, indoles cognoſcetur. Si functio quaecunque V duarum variabilium x & y differentietur poſita ſola x variabili, hocque differentiale denuo differentietur poſita ſola y variabili, tum poſt du-

duplicem hanc differentiationem idem prodibit, ac ſi ordine inverſo functio V primum poſita ſola y variabili differentiaretur, hocque differentiale poſita ſola x variabili denuo differentiaretur: utroque ſcilicet caſu prodibit eadem expreſſio huius formae $Zdxdy$. Ratio huius identitatis ex praecedente proprietate manifeſto ſequitur: ſi enim V differentietur poſita ſola x variabili, prodit Pdx; &, ſi V differentietur poſita ſola y variabili, prodit Qdy, horum differentialium vero differentialia modo indicato ſumta inter ſe aequalia eſſe, ante demonſtravimus. Ceterum haec indoles immediate ſequitur ex ratiocinio (§. 227) allato.

231. Relatio inter P & Q, ſi $Pdx + Qdy$ fuerit differentiale functionis V ſequenti etiam modo indicari poteſt. Quoniam P & Q ſunt functiones ipſarum x & y, differentientur ambae poſita utraque x & y variabili:

Si ſcilicet fuerit $dV = Pdx + Qdy$

fit $dP = pdx + rdy$

& $dQ = qdx + sdy$.

Poſito ergo x conſtante erit

$dP = rdy$, & $d.Pdx = rdxdy$.

Deinde poſito y conſtante erit

$dQ = qdx$, & $d.Qdy = qdxdy$.

Cum igitur haec duo differentialia $rdxdy$ & $qdxdy$ ſint inter ſe aequalia, ſequitur fore $q = r$. Functiones ergo P & Q ita invicem connectuntur, ut ſi ambae differentientur, uti fecimus, quantitates q & r inter ſe fiant aequales. Brevitatis gratia autem hoc ſaltem capite quantitates r & q ita commode denotari ſolent, ut r indicetur per $\left(\frac{dP}{dy}\right)$, qua ſcriptura deſignatur P ita differentiari, ut ſola y tanquam variabilis tractetur, atque differentiale iſtud per dy dividatur: ſic enim prodibit quantitas finita r. Simili modo ſignificabit $\left(\frac{dQ}{dx}\right)$ quan-

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quantitatem finitam q, quia hac ratione indicatur functionem Q ſola x poſita variabili differentiari, tumque differentiale per dx dividi debere.

232. Utamur ergo hoc ſcribendi modo, etiamſi alias ambiguitatem afferre poſſit, quae tamen hic per clauſulas evitatur, ut ambages in deſcribendis differentiandi conditionibus evitemus, ſicque breviter relationem inter P & Q ita verbis exprimere poterimus, ut dicamus eſſe $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$. In huiuſmodi ſcilicet fractionibus denominator praeter propriam ſignificationem, qua numerator per eum dividi debet, indicat numeratoris differentiale ita eſſe capiendum, ut ea ſola quantitas cuius differentiale denominatorem conſtituit, tanquam variabilis ſpectetur. Hoc enim modo per diviſionem differentialia prorſus ex calculo egredientur, iſtaeque fractiones $\left(\frac{dP}{dy}\right)$ & $\left(\frac{dQ}{dx}\right)$ exhibebunt quantitates finitas, quae in praeſenti caſu erunt inter ſe aequales. Hoc itaque modo recepto quantitates quoque p & s ita denotare licebit, ut ſit $p = \left(\frac{dP}{dx}\right)$ & $s = \left(\frac{dQ}{dy}\right)$, ſi quidem ut monitum eſt, differentiatio numeratoris per denominatorem reſtringatur.

233. Conſentit haec proprietas mirifice cum proprietate, quam ante in functionibus homogeneis ineſſe oſtendimus. Sit enim V functio homogenea n dimenſionum ipſarum x & y, ponaturque $dV = Pdx + Qdy$, atque demonſtravimus fore $nV = Px + Qy$, ideoque

$$Q = \frac{nV}{y} - \frac{Px}{y}. \quad \text{Sit} \quad dP = pdx + rdy;$$

eritque $\left(\frac{dP}{dy}\right) = r$, cui aequale eſſe $\left(\frac{dQ}{dx}\right)$ ita oſtendetur. Differentietur Q poſita ſola x variabili, & quia in hac

hac hypotheſi eſt $dQ = \frac{nPdx}{y} - \frac{Pdx}{y} - \frac{xpdx}{y}$, fiet $\left(\frac{dQ}{dx}\right) = \frac{(n-1)P}{y} - \frac{px}{y}$, debebitque eſſe $\frac{(n-1)P}{y} - \frac{px}{y} = r$ ſeu $(n-1)P = px + ry$. Quae aequalitas inde fit perſpicua, quod P ſit functio homogenea $n-1$ dimenſionum ipſarum x & y, unde eius differentiale $dP = pdx + rdy$, ob proprietatem functionum homogenearum, ita debet eſſe comparatum, ut ſit $(n-1)P = px + ry$.

234. Iſta proprietas, quod ſit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$, quam omnibus functionibus duarum variabilium x & y communem eſſe oſtendimus, nobis quoque patefaciet naturam functionum trium pluriumve variabilium. Sit V functio quaecunque trium variabilium x, y, & z, ac ponatur $dV = Pdx + Qdy + Rdz$. Quod ſi igitur in hac differentiatione z tanquam conſtans tractaretur, foret utique $dV = Pdx + Qdy$; hoc autem caſu per antecedentia debet eſſe $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$. Deinde ſi quantitas y conſtans aſſumeretur, foret $dV = Pdx + Rdz$, erit ergo $\left(\frac{dP}{dz}\right) = \left(\frac{dR}{dx}\right)$. Denique poſito x conſtante reperietur $\left(\frac{dQ}{dz}\right) = \left(\frac{dR}{dy}\right)$. In differentiali ergo $Pdx + Qdy + Rdz$ functionis V quantitates P, Q, & R ita a ſe invicem pendent, ut ſit

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right);\ \left(\frac{dP}{dz}\right) = \left(\frac{dR}{dx}\right);\ \&\ \left(\frac{dQ}{dz}\right) = \left(\frac{dR}{dy}\right).$$

235. Sequitur hinc iſta functionum, quae tres pluresve va-

variabiles involvunt, proprietas analoga ei, quam ſupra (230) de functionibus duarum variabilium oſtendimus. Si fuerit V functio quaecunque trium variabilium x, y, & z, eaque continuo ter differentietur, ita ut primum una quantitatum, puta x, ſola variabilis ponatur, in differentiatione ſecunda ſola y, atque in tertia ſola z variabilis aſſumatur, prodibit expreſſio huius formae $Zdxdydz$, quae eadem reperietur, quocunque alio ordine quantitates x, y, & z collocentur. Sex igitur diverſis modis poſt triplicem differentiationem ad eandem expreſſionem $Zdxdydz$ pervenietur, quoniam ordo quantitatum x, y, & z ſexies variari poteſt. Quicunque ergo ordo eligatur, ſi functio V differentietur poſita ſola prima variabili, hocque differentiale denuo differentietur poſita ſola ſecunda variabili, atque differentiale hoc iterum differentietur poſita ſola tertia variabili, eadem prodibit expreſſio, utcunque ordo quantitatum x, y, & z varietur.

236. Quo ratio huius proprietatis clarius perſpiciatur, ponamus eſſe $dV = Pdx + Qdy + Rdz$; deinde etiam quantitates P, Q, & R differentiemus, eruntque earum differentialia per ante demonſtrata ita comparata:

$$dP = pdx + sdy + tdz$$
$$dQ = sdx + qdy + udz$$
$$dR = tdx + udy + rdz.$$

Differentietur nunc V poſito ſolo x variabili prodibit Pdx, quod differentiale iterum differentietur poſito ſolo y variabili atque habebitur $sdxdy$; quod ſi differentietur poſito ſolo z variabili, poſtquam per $dxdydz$ fuerit diviſum, obtinebitur $\left(\frac{ds}{dz}\right)$. Collocentur nunc variabiles hoc ordine y, x, z, atque prima differentiatio dabit Qdy, ſecunda $sdxdy$, & tertia (facta diviſione per $dxdydz$) dabit $\left(\frac{ds}{dz}\right)$ ut ante. Diſponantur variabiles hoc ordine z, y, x, ac prima differen-

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tiatio dabit Rdz ſecunda $udydz$, tertia vero poſt diviſionem per $dxdydz$ praebet $\left(\frac{du}{dx}\right)$. At cum poſito y conſtante ſit $dQ = sdx + udz$; erit $\left(\frac{ds}{dz}\right) = \left(\frac{du}{dx}\right)$, uti pariter eſt demonſtratum.

237. Ponamus eſſe $V = \frac{xxy}{aa - zz}$; hancque functionem toties ter differentiemus, quoties ordo variabilium x, y, z variari poteſt:

	I. DIFFER.		II. DIFFER.		III. DIFFER.
poſito variabili	ſolo x $\frac{2xydx}{aa - zz}$	;	ſolo y $\frac{2xdxdy}{aa - zz}$	;	ſolo z $\frac{4xzdxdydz}{(aa - zz)^2}$
poſito variabili	ſolo x $\frac{2xydx}{aa - zz}$	;	ſolo z $\frac{4xyzdxdz}{(aa - zz)^2}$	;	ſolo y $\frac{4xzdxdydz}{(aa - zz)^2}$
poſito variabili	ſolo y $\frac{xxdy}{aa - zz}$	;	ſolo x $\frac{2xdxdy}{aa - zz}$	;	ſolo z $\frac{4xzdxdydz}{(aa - zz)^2}$
poſito variabili	ſolo y $\frac{xxdy}{aa - zz}$	;	ſolo z $\frac{2xxzdydz}{(aa - zz)^2}$	;	ſolo x $\frac{4xzdxdydz}{(aa - zz)^2}$
poſito variabili	ſolo z $\frac{2xxyzdz}{(aa - zz)^2}$	;	ſolo x $\frac{4xyzdxdz}{(aa - zz)^2}$	,	ſolo y $\frac{4xzdxdydz}{(aa - zz)^2}$
poſito variabili	ſolo z $\frac{2xxyzdz}{(aa - zz)^2}$	;	ſolo y $\frac{2xxzdydz}{(aa - zz)^2}$	;	ſolo x $\frac{4xzdxdydz}{(aa - zz)^2}$

ex quo exemplo patet, quocunque ordine tres variabiles fuerint

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rint aſſumtae, poſt triplicem differentiationem ſemper eandem prodire expreſſionem $\frac{4xzdxdydz}{(aa-zz)^2}$.

238. Uti autem poſt triplicem differentiationem ad eandem expreſſionem eſt perventum, ita quoque conſenſus deprehenditur in differentialibus, quae ſecunda differentiatio ſuppeditavit. In iis ſcilicet expreſſio quaevis bis occurrit; unde patet, quae formulae iiſdem differentialibus ſint affectae, eaſdem quoque inter ſe eſſe aequales, atque differentialia tertia ideo eſſe omnia inter ſe aequalia, quia iiſdem differentialibus $dxdydz$ ſunt affecta. Hinc igitur concludimus, ſi V fuerit functio quotcunque variabilium x, y, z, v, u, &c. eaque ſucceſſive aliquoties differentietur, ut ſemper unica tantum quantitas variabilis aſſumatur; tum quoties ad expreſſiones perveniatur, quae iiſdem differentialibus ſint affectae, eas quoque inter ſe aequales fore. Sic duplici differentiatione orietur huiuſmodi expreſſio $Zdxdy$, dum in altera ſola x, in altera ſola y aſſumta eſt variabilis: perindeque eſt utra prius, poſteriuſve ſit variabilis aſſumta. Simili modo ſex variis modis per triplicem differentiationem eadem exſurget expreſſio $Zdxdydz$; atque viginti quatuor variis modis pervenietur poſt quadruplicem differentiationem ad eandem expreſſionem huius formae $Zdxdydzdv$, atque ita porro.

239. Veritatem horum Theorematum quilibet adhibita levi attentione ex ante explicatis principiis facile agnoſcet, atque propria meditatione facilius intuebitur, quam tantis verborum ambagibus, ſine quibus demonſtrationes proferri non poſſent. Quia vero harum proprietatum cognitio maximi eſt momenti in calculo integrali, Tyrones ſunt monendi, ut non ſolum has proprietates ipſi diligenter meditentur, earumque veritatem ſcrutentur, ſed etiam pluribus exemplis comprobent; quo hoc pacto ſibi hanc materiam familiariorem reddant, fructuſque inde natos poſtmodum facilius percipere que-

queant. Neque vero ſolum tyrones, ſed etiam ii, qui principiis calculi differentialis iam ſunt imbuti, ad hoc ſunt cohortandi; quoniam in omnibus fere manuductionibus ad hanc Analyſeos partem hoc argumentum penitus praetermitti ſolet. Plerumque enim Auctores ſolas differentiationis regulas praeſcripſiſſe, earumque uſum in Geometria ſublimiori oſtendiſſe fuerunt contenti, neque in naturam atque proprietates differentialium inquiſiverunt; unde tamen maxima ſubſidia in calculum integralem redundant. Quam ob cauſam hoc argumentum fere novum in iſto Capite fuſius perſequi viſum eſt, quo ſimul via ad integrationes alias difficiliores pararetur, atque negotium poſtea ſuſcipiendum ſublevaretur.

240. Cognitis igitur his proprietatibus, quibus differentialia functionum duas pluresve variabiles involventium gaudent, facile poterimus dignoſcere, utrum formula differentialis propoſita, in qua occurrunt duae pluresve variabiles, ſit orta ex differentiatione cuiuſpiam functionis finitae an ſecus. Si enim in formula $Pdx + Qdy$ non fuerit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$, certo poterimus affirmare, nullam exiſtere functionem ipſarum x & y, cuius differentiale ſit $= Pdx + Qdy$: neque ergo infra in calculo integrali huiuſmodi formulae integrale indagari poteſt. Sic cum in $yxdx + xxdy$ requiſita conditio non adſit, nulla datur functio, cuius differentiale eſt $= yxdx + xxdy$. Utrum autem ſemper, quoties eſt $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$ formula ex differentiatione cuiuſpiam functionis ſit orta, quaeſtio eſt, quae demum ex integrationis principiis ſolide affirmari poterit.

241. Si in formula differentiali propoſita tres pluresve inſint variabiles, uti $Pdx + Qdy + Rdz$; tum ea ex differentiatione ortum traxiſſe omnino nequit, niſi tres iſtae conditiones in ea locum habeant, ut ſit

dP

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right); \left(\frac{dP}{dz}\right) = \left(\frac{dR}{dx}\right); \& \left(\frac{dQ}{dz}\right) = \left(\frac{dR}{dy}\right),$$

Quarum conditionum, ſi una tantum deſit, certo affirmare debemus, nullam extare functionem ipſarum x, y, & z, cuius differentiale ſit $Pdx + Qdy + Rdz$; huius modi ergo formularum differentialium nequidem requiri poſſunt integralia, hincque integrationem prorſus non recipere dicuntur. Facile autem intelligitur in calculo integrali formulas differentiales ante dignoſci oportere, utrum integrationis ſint capaces, quam inveſtigatio integralis actu ſuſcipiatur.

CA-

CAPUT VIII.

DE FORMULARUM DIFFERENTIALIUM ULTERIORI DIFFERENTIATIONE.

242.

Si unica variabilis adſit, eiuſque differentiale primum conſtans aſſumatur, ſupra iam methodus eſt tradita differentialia cuiuſque gradus inveniendi. Scilicet ſi functionis cuiuſvis differentiale denuo differentietur, oritur eius differentiale ſecundum, hocque iterum differentiatum dat functionis differentiale tertium, atque ita porro. Haec vero eadem regula locum quoque habet, ſive functio plures involvat variabiles ſive unicam tantum, cuius differentiale primum non ponitur conſtans. Sit igitur V functio quaecunque ipſius x, neque vero dx ſit conſtans, ſed utcunque variabile, ita ut ipſius dx differentiale ſit $= ddx$, huiuſque differentiale $= d^3x$, & ita porro, atque inveſtigemus differentialia ſecundum & ſequentia functionis V.

243. Ponamus differentiale primum functionis V eſſe $= Pdx$, ubi erit P functio quaepiam ipſius x pendens ab V. Si iam functionis V differentiale ſecundum invenire velimus, eius differentiale primum Pdx denuo differentiari oportet; quod cum ſit productum ex duabus quantitatibus variabilibus P & dx, quarum illius differentiale ſit $dP = pdx$, huius vero dx differentiale ddx, per regulam de factoribus datam erit differentiale ſecundum $ddV = Pddx + pdx^2$. Deinde ſi ponatur $dp = qdx$, cum differentiale ipſius dx^2 ſit $2dxddx$, erit iterum differentiando

$$d^3V = Pd^3x + dPddx + 2pdxddx + dpdx^2,$$

iam ob $dP = pdx$ & $dp = qdx$; erit

$$d^3V = Pd^3x + 3pdxddx + qdx^3,$$

ſimilique modo ulteriora differentialia invenientur.

244. Applicemus haec ad poteſtates ipſius x, quarum ſingula differentialia inveſtigemus, ſi dx non ponatur conſtans:

I. Sit igitur $V = x$; erit $dV = dx$; $d^2V = d^2x$; $d^3V = d^3x$; $d^4V = d^4x$;

&c.

II. Sit $V = x^2$; erit $dV = 2xdx$; &

$$ddV = 2xddx + 2dx^2$$
$$d^3V = 2xd^3x + 6dxddx$$
$$d^4V = 2xd^4x + 8dxd^3x + 6ddx^2$$
$$d^5V = 2xd^5x + 10dxd^4x + 20ddxd^3x$$

&c.

III. Si in genere fuerit $V = x^n$; erit

$$dV = nx^{n-1}dx$$
$$ddV = nx^{n-1}ddx + n(n-1)x^{n-2}dx^2$$
$$d^3V = nx^{n-1}d^3x + 3n(n-1)x^{n-2}dxddx + n(n-1)(n-2)x^{n-3}dx^3$$
$$d^4V = nx^{n-1}d^4x + 4n(n-1)x^{n-2}dxd^3x + 3n(n-1)x^{n-2}ddx^2 + 6n(n-1)(n-2)x^{n-3}dx^2ddx + n(n-1)(n-2)(n-3)x^{n-4}dx^4$$

&c.

Si igitur fuerit dx conſtans, ac propterea

$$ddx = 0,\quad d^3x = 0,\quad d^4x = 0,\quad \&c.$$

orientur eadem differentialia, quae iam ſupra pro hac hypotheſi ſunt inventa.

245. Quoniam igitur differentialia cuiuſque ordinis ipſius x eadem lege differentiantur, qua quantitates finitae, expreſſiones quaecunque, in quibus praeter quantitatem finitam eius differentialia occurrunt, ſecundum praecepta ſupra data differentiari poterunt. Quam operationem, cum nonnunquam occurrat, hic aliquot exemplis illuſtrabimus.

I.

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I. Si fuerit $V = \frac{xddx}{dx^2}$, differentiando prodibit

$$dV = \frac{xd^3x}{dx^2} + \frac{ddx}{dx} - \frac{2xddx^2}{dx^3}.$$

II. Si fuerit $V = \frac{x}{dx}$; erit $dV = 1 - \frac{xddx}{dx^2}$,
ubi nihil impedit, quod pro V quantitatem infinite magnam poſuimus.

III. Si fuerit $V = xxl\frac{ddx}{dx^2}$, quia tranſmutatur

V in $xxlddx - 2xxldx$;

erit ſecundum regulas conſuetas differentiando:

$$dV = 2xdxlddx + \frac{xxd^3x}{ddx} - 4xdxldx - \frac{2xxddx}{dx}.$$

Simili autem modo differentialia altiora ipſius V reperientur.

246. Si expreſſio propoſita duas variabiles involvat, nempe x & y, vel unius differentiale ponitur conſtans vel neutrius; arbitrarium enim eſt alterutrius differentiale conſtans aſſumi, quia ab arbitrio noſtro pendet, quemadmodum unius valores ſucceſſivos creſcere ſtatuere velimus. Neque vero utriuſque variabilis differentialia ſimul ſtatui poſſunt conſtantia, hoc ipſo enim relatio inter variabiles x & y aſſumeretur, quae tamen vel nulla eſt, vel incognita ponitur. Si enim, dum x aequabiliter creſcere ponimus, y quoque aequalia incrementa capere ſtatueretur, tum eo ipſo indicaretur fore $y = ax + b$; ſicque y ab x penderet, quod tamen aſſumere non licet. Hancobrem vel unius tantum variabilis differentiale conſtans aſſumi poteſt vel nullum. Quodſi autem differentiationes abſolvere noverimus nullo differentiali aſſumto conſtante, ſimul quoque differentialia conſtabunt, ſi alterutrum differentiale ponatur conſtans: tantum enim opus eſt, ut ſi

dx

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dx conſtans ſtatuatur, ubique termini continentes ddx, d^3x, d^4x, &c. deleantur.

247. Denotet ergo V functionem quamcunque finitam ipſarum x & y, ſitque $dV = Pdx + Qdy$. Ad differentiale ipſius V ſecundum inveniendum aſſumamus utrumque differentiale dx & dy variabile, & cum P & Q ſint functiones ipſarum x & y ſtatuamus:

$$dP = pdx + rdy$$
$$dQ = rdx + qdy$$

ſupra enim vidimus eſſe $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right) = r$.

His poſitis differentietur $dV = Pdx + Qdy$, & reperietur:

$$ddV = Pddx + pdx^2 + 2rdxdy + Qddy + qdy^2.$$

Si igitur differentiale dx ſtatuatur conſtans, erit

$$ddV = pdx^2 + 2rdxdy + Qddy + qdy^2,$$

ſin autem differentiale dy ſtatueretur conſtans, foret

$$ddV = Pddx + pdx^2 + 2rdxdy + qdy^2.$$

248. Si igitur functio quaecunque ipſarum x & y bis differentietur, nullo differentiali poſito conſtante, eius differentiale ſecundum ſemper huiuſmodi formam habebit:

$$ddV = Pddx + Qddy + Rdx^2 + Sdy^2 + Tdxdy$$

pendebunt autem quantitates P, Q, R, S, & T ita a ſe invicem, ut ſit ſignandi modo Capite praecedente adhibito:

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right);\ R = \left(\frac{dP}{dx}\right);\ S = \left(\frac{dQ}{dy}\right);$$
$$\&\ \ T = 2\left(\frac{dQ}{dx}\right) = 2\left(\frac{dP}{dy}\right),$$

quarum conditionum ſi vel unica deſit, certo affirmare poterimus, formulam propoſitam nullius functionis eſſe differentiale ſecundum. Statim ergo dignoſci poterit, utrum huiuſmodi formula ſit cuiuſpiam quantitatis differentiale ſecundum an minus.

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249. Simili modo differentialia tertia ac ſequentia inveniuntur, quod in exemplo particulari oſtendiſſe expediet, quam formulas generales adhibendo.

Sit igitur $V = xy$;

Erit $dV = ydx + xdy$

$ddV = yddx + 2dxdy + xddy$

$d^3V = yd^3x + 3dyddx + 3dxddy + xd^3y$

$d^4V = yd^4x + 4dyd^3x + 6ddxddy + 4dxd^3y + xd^4y$

&c.

in quo exemplo coefficientes numerici legem poteſtatum binomii ſequuntur, indeque quouſque libuerit continuari poſſunt.

At ſi fuerit $V = \frac{y}{x}$;

$$\text{Erit } dV = \frac{dy}{x} - \frac{ydx}{xx}$$

$$ddV = \frac{ddy}{x} - \frac{2dxdy}{xx} + \frac{2ydx^2}{x^3} - \frac{yddx}{x^2}$$

$$d^3V = \frac{d^3y}{x} - \frac{3dxddy}{xx} + \frac{6dx^2dy}{x^3} - \frac{3dyddx}{x^2} + \frac{6ydxddx}{x^3} - \frac{6ydx^3}{x^4} - \frac{yd^3x}{x^2}$$

in quo exemplo progreſſio differentialium non tam facile patet quam in praecedente.

250. Neque vero tantum haec differentiandi methodus ad functiones finitas adſtringitur, ſed etiam eodem negotio cuiuſvis expreſſionis, quae iam differentialia in ſe continet, differentiale inveniri poteſt, ſive unum quoddam differentiale aſſumitur conſtans ſive minus. Cum enim ſingula differentialia aeque & eadem lege differentientur ac quantitates finitae, regulae in praecedentibus capitibus traditae, etiam hic valent atque obſervari debent. Denotet igitur V eam expreſſionem, quam

quam differentiari oportet, ſive ſit finita, ſive infinite magna ſive infinite parva; atque ratio differentiationis ex his exemplis perſpicietur:

I. Sit $V = \sqrt{(dx^2 + dy^2)}$;

Erit $dV = \frac{dxddx + dyddy}{\sqrt{(dx^2 + dy^2)}}$.

II. Sit $V = \frac{ydx}{dy}$;

Erit $dV = dx + \frac{yddx}{dy} - \frac{ydxddy}{dy^2}$.

III. Sit $V = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dxddy - dyddx}$;

Erit $dV = \frac{(3dxddx + 3dyddy)\sqrt{(dx^2 + dy^2)}}{dxddy - dyddx}$

$$- \frac{(dx^2 + dy^2)^{\frac{3}{2}}(dxd^3y - dyd^3x)}{(dxddy - dyddx)^2}.$$

quae differentialia cum ſint generaliſſime ſumta, nullo differentiali pro conſtante habito, hinc facile ea differentialia derivari poterunt, quae oriuntur, ſi vel dx vel dy ſtatuatur conſtans.

251. Quia nullo differentiali conſtante aſſumto, nulla etiam lex, ſecundum quam ſucceſſivi variabilium valores progrediantur, praeſcribitur, differentialia ſecunda & ſequentium ordinum non erunt determinata, neque quicquam certi ſignificabunt. Hinc formula, in qua differentialia ſecunda atque altiora continentur, nullum determinatum habebit valorem, niſi quodpiam differentiale conſtans ſit aſſumtum; ſed eius ſignificatio erit vaga, atque variabitur, prouti aliud atque aliud differentiale fuerit conſtans poſitum. Interim tamen dantur quo-

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que eiuſmodi expreſſiones differentialia ſecunda continentes, quae, etiamſi nullum differentiale poſitum ſit conſtans, tamen ſignificatum determinatum complectuntur, qui perpetuo idem maneat, quodcunque differentiale conſtans ſtatuatur. Huiuſmodi autem formularum naturam infra diligentius ſcrutabimur, modumque trademus eas ab aliis, quae valores determinatos non includunt, dignoſcendi.

252. Quo haec ratio formularum, in quibus differentialia ſecunda vel altiora inſunt, facilius perſpiciatur, contemplemur primum formulas unicam variabilem continentes, atque facile patet, ſi in quapiam formula inſit eius variabilis x differentiale ſecundum ddx, nullumque differentiale conſtans ſtatuatur, formulam nullum valorem fixum habere poſſe. Si enim ſtatuatur differentiale ipſius x conſtans, fiet $ddx = 0$; ſin autem ipſius xx differentiale $2xdx$ ſeu xdx conſtans ponatur, cum ipſius xdx differentiale $xddx + dx^2$ ſit $= 0$, fiet $ddx = -\frac{dx^2}{x}$. Verum ſi poteſtatis cuiuſcunque x^n differentiale $nx^{n-1}dx$ ſeu $x^{n-1}dx$ debeat eſſe conſtans; erit eius differentiale ſecundum $x^{n-1}ddx + (n-1)x^{n-2}dx^2 = 0$, ideoque $ddx = -\frac{(n-1)dx^2}{x}$. Alii valores pro ddx prodibunt, ſi aliarum ipſius x functionum differentialia conſtantia ponantur. Manifeſtum autem eſt, formulam, in qua ddx occurrat, diverſiſſimos induere valores, prout loco ddx ſcribatur vel 0 vel $-\frac{dx^2}{x}$ vel $-\frac{(n-1)dx^2}{x}$ vel alia huiuſmodi expreſſio. Scilicet ſi proponatur formula $\frac{xxddx}{dx^2}$, quae ob ddx & dx^2 infinite parva homogenea, finitum valorem habere deberet; ea poſito dx conſtante abit in 0, ſi ſit dx^2 conſtans, ea abit in $-x$; ſi ſit dx^3 conſtans, ea abit in $-2x$; ſi dx^4 ſit conſtans, ea abit in $-3x$ & ita por-

porro. Neque ergo determinatum valorem habere poteſt, niſi definiatur, cuiuſmodi differentiale conſtans ſit aſſumtum.

253. Iſta inconſtantia ſignificationis ſimili ratione oſtenditur, ſi differentiale tertium in quapiam formula inſit. Conſideremus hanc formulam $\frac{x^3 d^3 x}{dxddx}$, quae pariter finitum valorem prae ſe fert. Si differentiale dx ſit conſtans, abit ea in $\frac{0}{0}$, cuius valor mox patebit. Sit dx^2 conſtans, erit

$ddx = -\frac{dx^2}{x}$; & denuo differentiando

$$d^3 x = -\frac{2dxddx}{x} + \frac{dx^3}{x^2} = \frac{3dx^3}{x^2}, \quad \text{ob } ddx = -\frac{dx^2}{x},$$

hoc ergo caſu formula propoſita $\frac{x^3 d^3 x}{dxddx}$ abit in $-3x^2$.

At ſi fuerit dx^n conſtans, erit $ddx = -\frac{(n-1)dx^2}{x}$, hincque

$$d^3 x = -\frac{2(n-1)dxddx}{x} + \frac{(n-1)dx^3}{x^2} = \frac{2(n-1)^2 dx^3}{xx} + \frac{(n-1)dx^3}{xx} = \frac{(2n-1)(n-1)dx^3}{xx}$$

Hoc ergo caſu erit

$$\frac{d^3 x}{ddx} = -\frac{(2n-1)dx}{x}, \& \frac{x^3 d^3 x}{dxddx} = -(2n-1)x^2,$$

unde patet ſi ſit $n = 1$, ſeu dx conſtans, valorem formulae fore $= -x^2$. Ex quo manifeſtum eſt, ſi in quapiam formula differentialia tertia vel altiora occurrant, neque ſimul indicetur, cuiuſmodi differentiale aſſumtum ſit conſtans, eam formulam nullum certum valorem habere, atque adeo nihil

pror-

prorſus ſignificare; quamobrem tales expreſſiones in calculo occurrere non poſſunt.

254. Simili modo ſi formula contineat duas pluresve variabiles, in eaque occurrant differentialia ſecundi altiorisve gradus, intelligetur valorem determinatum locum habere non poſſe, niſi differentiale quodpiam conſtans ſtatuatur, iis tantum exceptis caſibus, quos mox perpendemus. Quum primum enim ddx in quapiam formula ineſt, quoniam pro variis differentialibus, quae conſtantia ponuntur, valor ipſius ddx perpetuo variatur, fieri nequit, ut formula ſtatum obtineat valorem; hocque idem valet de quovis differentiali altiori ipſius x, atque etiam de differentialibus reliquarum variabilium ſecundis & altioribus. Sin autem duarum pluriumve variabilium differentialia ſecunda inſint, fieri poteſt, ut inconſtantia ab uno orienda per inconſtantiam reliquarum deſtruatur; hincque naſcitur ille caſus, cuius meminimus, quo formula huiuſmodi differentialia ſecunda duarum pluriumve variabilium involvens valorem definitum habere poteſt, non obſtante quod nullum differentiale conſtans ſit poſitum.

255. Haec igitur formula $\frac{yddx + xddy}{dxdy}$, ſtatam atque fixam ſignificationem habere nequit, niſi quodpiam differentiale primum conſtans ſtatuatur. Nam ſi dx conſtans ponatur habebitur $\frac{xddy}{dxdy}$; ſin autem dy conſtans ponatur, habebitur $\frac{yddx}{dxdy}$; manifeſtum autem eſt has formulas non neceſſario inter ſe eſſe aequales. Si enim neceſſario eſſent aequales, tales manere deberent, quaecunque functio ipſius x loco y ſubſtitueretur. Ponamus tantum eſſe $y = xx$, & cum poſito dx conſtante, ſit $ddy = 2dx^2$, formula $\frac{xddy}{dxdy}$ abibit

in

in 1, ſin autem dy ſeu $2xdx$ ponatur conſtans, fiet $ddy = 2xddx + 2dx^2 = 0$, ideoque $ddx = -\frac{dx^2}{x}$, unde formula $\frac{yddx}{dxdy}$ abit in $-\frac{1}{2}$. Cum igitur in unico caſu reperiatur diſcrepantia, multo minus in genere erit $\frac{xddy}{dxdy}$ poſito dx conſtante aequalis $\frac{yddx}{dxdy}$ poſito dy conſtante. Deinde quia formula $\frac{yddx + xddy}{dxdy}$ ſibi non conſtat, dummodo vel dx vel dy conſtans ponatur, multo minus ſibi conſtabit, ſi functionis cuiuſvis vel ipſius x vel ipſius y vel utriuſque differentiale conſtans ponatur.

256. Hinc apparet huiuſmodi formulam ſtatum valorem habere non poſſe, niſi ita ſit comparata, ut poſtquam loco variabilium y, & z, quae praeter x inſunt, functiones quaecunque ipſius x fuerint ſubſtitutae, differentialia ſecunda & altiora ipſius x, nempe ddx, d^3x, &c. penitus ex calculo excedant. Si enim poſt talem ſubſtitutionem quancunque in formula adhuc relinqueretur ddx, vel d^3x, vel d^4x, &c. quia haec differentialia, prout alia aliaque conſtantia aſſumuntur, ſignificationem ſuam variant, valor quoque ipſius formulae erit vagus. Sic comparata eſt formula ante propoſita $\frac{yddx + xddy}{dxdy}$, quae ſi ſtatum haberet valorem, quicquid y ſignificet, ſtatum quoque habere deberet valorem, ſi y denotaret funtionem quampiam ipſius x. At ſi tantum ponamus $y = x$, formula abit in $\frac{2xddx}{dx^2}$ quae utique ob ddx in ea contentum eſt vaga, atque alios alioſque valores induit, prouti alia atque

atque alia differentialia conſtantia ponuntur, uti ex §. 252 ſatis eſt manifeſtum.

257. Dubium autem hic ſubnaſcetur, utrum dentur tales formulae duo plurave differentialia ſecundi altiorisve gradus continentia, quae hac proprietate gaudeant, ut ſi loco reliquarum variabilium quaecunque functiones unius ſubſtituantur, differentialia ſecundi gradus prorſus ſe deſtruant. Huic dubio primum ita occurramus, ut huiuſmodi formulam proponamus, quae iſta proprietate ſit praedita, quo per explorationem vis quaeſtionis melius percipiatur. Dico igitur hanc formulam $\frac{dyddx - dxddy}{dx^3}$ memoratam proprietatem poſſidere: quaecunque enim functio ipſius x loco y ſubſtituatur, ſemper differentialia ſecundi gradus penitus evaneſcent; quam proprietatem ſequentibus exemplis comprobemus.

I. Sit $y = x^2$; erit $dy = 2xdx$, & $ddy = 2xddx + 2dx^2$, qui valores in formula $\frac{dyddx - dxddy}{dx^3}$ ſubſtituti dabunt,

$$\frac{2xdxddx - 2xdxddx - 2dx^3}{dx^3} = -2.$$

II. Sit $y = x^n$; erit $dy = nx^{n-1}dx$,
& $ddy = nx^{n-1}ddx + n(n-1)x^{n-2}dx^2$

qui valores ſubſtituti formulam $\frac{dyddx - dxddy}{dx^3}$ tranſmutabunt in hanc

$$\frac{nx^{n-1}dxddx - nx^{n-1}dxddx - n(n-1)x^{n-2}dx^3}{dx^3} = -n(n-1)x^{n-2}.$$

III. Sit $y = -\surd(1-xx)$; erit $dy = \frac{xdx}{\surd(1-xx)}$,

&

& $ddy = \frac{xddx}{\sqrt{(1-xx)}} + \frac{dx^2}{(1-xx)^{\frac{3}{2}}};$

atque formula $\frac{dyddx - dxddy}{dx^3}$, abit in

$$\frac{xddx}{dx^2\sqrt{(1-xx)}} - \frac{xddx}{dx^2\sqrt{(1-xx)}} - \frac{1}{(1-xx)^{\frac{3}{2}}} = \frac{-1}{(1-xx)^{\frac{3}{2}}}.$$

In his igitur omnibus exemplis differentialia ſecunda ddx ſe mutuo tollunt, hocque ita eveniet, quaecunque aliae functiones loco y ſubſtituantur.

258. Cum iſta exempla iam probaverint veritatem noſtrae propoſitionis, quod formula $\frac{dyddx - dxddy}{dx^3}$ fixum habeat valorem, etiamſi nullum differentiale conſtans ſit aſſumtum, demonſtrationem eo facilius adornare poterimus. Sit y functio quaecunque ipſius x, eiuſque differentiale dy huiuſmodi erit, ut ſit $dy = pdx$, atque p erit functio quaepiam ipſius x, eiuſque differentiale propterea huiuſmodi formam habebit $dp = qdx$, eritque q iterum functio ipſius x. Cum igitur ſit $dy = pdx$, erit differentiando $ddy = pddx + qdx^2$, & $dyddx - dxddy = pdxddx - pdxddx - qdx^3 = -qdx^3$; in qua expreſſione cum nullum inſit differentiale ſecundum, habebit ea valorem fixum, atque $\frac{dyddx - dxddy}{dx^3}$ erit $= -q$. Quomodocunque igitur y pendeat ab x, differentialia ſecunda in hac formula ſemper ſe mutuo tollent, hancque ob cauſam eius valor, qui alioquin eſſet vagus, fiet ſtatus ac fixus.

259. Quanquam hic poſuimus y eſſe functionem ipſius x, tamen veritas aeque ſubſiſtit, ſi y ab x prorſus non pendeat, uti aſſumſimus. Dum enim pro y functionem quamcunque ſubſtituimus, neque qualis ſit determinavimus, nullam pendentiam ab x ipſi y tribuimus. Interim tamen ſine fun-

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ctionis mentione demonſtratio formari poteſt; quaecunque enim y ſit quantitas ſive pendens ab x ſive non pendens, eius differentiale dy homogeneum erit cum dx, ſicque $\frac{dy}{dx}$ quantitatem finitam denotabit, cuius differentiale, quod capit, dum x in $x+dx$ & y in $y+dy$ abit, erit fixum, neque a differentialium ſecundorum lege pendebit. Sit igitur $\frac{dy}{dx}=p$;

erit $dy=pdx$, & $ddy=pddx+dpdx$, unde fit
$$dxddy-dyddx=dpdx^2,$$
cuius valor non eſt vagus, quia tantum differentialia prima continet; ac propterea idem manet, ſive quodpiam differentiale conſtans accipiatur, qualecunque id demum ſit, ſive nullum differentiale poſitum ſit conſtans.

260. Quia igitur $dyddx-dxddy$ non obſtantibus differentialibus ſecundis, quae potentia ſe mutuo deſtruere cenſeri poſſunt, ſignificationem habet fixam; expreſſio quaecunque, in qua nulla alia differentialia ſecunda praeter formulam $dyddx-dxddy$ inſunt, pariter ſignificationem habebit fixam. Seu ſi ponatur $dyddx-dxddy=\omega$, atque V fuerit quantitas ex x, y, earum differentialibus primis dx, dy atque ex ω utcunque compoſita, ea valorem habebit fixum. Cum enim in differentialibus primis dx & dy nulla ratio habeatur eius legis arbitrariae, qua valores ſucceſſivi ipſius x creſcere ponuntur, in ω differentialia ſecunda ſe mutuo tollunt, etiam ipſa quantitas V non erit vaga ſed fixa. Sic iſta expreſſio $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxddy-dyddx}$ valorem obtinet fixum, quamvis ea differentialibus ſecundis inquinata videatur, atque inſuper, quia numerator eſt homogeneus denominatori, valorem obtinet finitum, niſi is caſu vel infinitae magnus vel infinitae parvus evadat.

261.

261. Quemadmodum formula $dxddy - dyddx$ valorem fixum habere oſtenſa eſt, ita quoque ſi tertia variabilis z accedat; hae formulae $dxddz - dzddx$ & $dyddz - dzddy$ valores fixos habebunt. Hinc expreſſiones, quas tres variabiles x, y, & z involvunt, ſi in eis nulla alia differentialia ſecunda occurrant, praeter haec aſſignata, tum perinde erunt fixae, ac ſi nulla plane differentialia ſecunda ineſſent. Ita haec expreſſio:

$$\frac{(dx^2 + dy^2 + dz^2)^{\frac{3}{2}}}{(dx + dz)ddy - (dy + dz)ddx + (dx - dy)ddz}$$

non obſtantibus differentialibus ſecundis, fixa gaudet ſignificatione. Similique modo formulae exhiberi poſſunt, plures variabiles continentes, in quibus differentialia ſecunda non impediunt, quominus earum ſignificatio ſit fixa.

262. Exceptis ergo huius generis formulis, quae differentialia ſecunda complectuntur, reliquae omnes ſignificationes habebunt vagas, neque propterea in calculo locum habere poſſunt, niſi quodpiam differentiale primum definiatur, quod conſtans ſit aſſumtum. Statim vero atque differentiale quodpiam primum conſtans aſſumitur, omnes expreſſiones quotcunque variabiles contineant, & cuiuſcunque ordinis differentialia poſt primum in eas ingrediuntur, fixas obtinebunt ſignificationes, neque amplius ex calculo excluduntur. Si enim verbi gratia dx aſſumtum ſit conſtans, ipſius x differentialia ſecunda & ſequentia evaneſcunt; & quaecunque functiones ipſius x loco reliquarum variabilium y, z, &c. ſubſtituantur, earum differentialia ſecunda per dx^2, tertia per dx^3, &c. determinabuntur, ſicque inconſtantia a differentialibus ſecundis oriunda tollitur. Idem evenit, ſi alius variabilis ſeu functionis cuiuſcunque differentiale primum conſtans ponatur.

263. Ex his igitur ſequitur differentialia ſecunda & altiorum ordinum revera nunquam in calculum ingredi, atque ob vagam ſignificationem prorſus ad Analyſin eſſe inepta.

Quando enim differentialia ſecunda adeſſe videntur, vel differentiale quodpiam primum conſtans aſſumitur, vel nullum. Priori caſu differentialia ſecunda prorſus ex calculo evaneſcunt, dum per differentialia prima determinantur. Poſteriori caſu autem niſi ſe mutuo deſtruant, ſignificatio erit vaga, & propterea in Analyſi locum nullum inveniunt; ſin autem ſe mutuo deſtruunt, tantum apparenter adſunt, & revera ſolae quantitates finitae cum ſuis differentialibus primis adeſſe cenſendae ſunt. Quoniam tamen ſaepiſſime apparenter tantum in calculo uſurpantur, neceſſe fuit, ut methodus eas tractandi exponeretur. Modum autem mox oſtendemus, cuius ope differentialia ſecunda & altiora ſemper exterminari queant.

264. Si expreſſio unicam contineat variabilem x, eiusque differentialia altiora ddx, d^3x, d^4x, &c. in ea occurrant, ea ſignificatum fixum habere nequit, niſi quodpiam differentiale primum conſtans ſit poſitum. Sit igitur t illa quantitas variabilis, cuius differentiale dt ſit conſtans poſitum, ita ut ſit $ddt = 0$, $d^3t = 0$, $d^4t = 0$, &c. Ponatur $dx = pdt$; eritque p quantitas finita, cuius differentiale vaga ſignificatione differentialium ſecundorum non afficietur, hincque etiam $\frac{dp}{dt}$ erit quantitas finita. Sit $dp = qdt$, ſimilique modo ulterius $dq = rdt$; $dr = sdt$; &c. erunt q, r, s, &c. quantitates finitae fixos ſignificatus habentes. Cum igitur ſit $dx = pdt$; erit

$$ddx = dpdt = qdt^2;\ d^3x = dqdt^2 = rdt^3;$$
$$d^4x = drdt^3 = sdt^4;\ \&c.$$

qui valores ſi loco ddx, d^3x, d^4x, &c. ſubſtituantur, tota expreſſio meras quantitates finitas cum differentiali primo dt continebit, ideoque non amplius vagam ſignificationem habebit.

265. Si x ſit functio ipſius t, poterit hoc modo quantitas x prorſus eliminari, ita ut ſola quantitas t cum ſuo differentiali dt in expreſſione remaneat: ſin autem t ſit functio

ctio ipſius x, viciſſim quoque x erit ipſius t functio. Interim tamen ipſa quantitas x cum ſuo differentiali primo dx, in calculo retineri poteſt, dummodo poſt ſubſtitutiones ante factas ubique loco t & dt earum valores per x & dx expreſſi reſtituantur. Quod quo planius fiat, ponamus t eſſe $= x^n$, ita ut differentiale primum ipſius x^n conſtans ſit poſitum.

Quia igitur eſt

$$dt = nx^{n-1}dx\ ; \quad \text{erit} \quad p = \frac{1}{nx^{n-1}}\ ; \quad \&$$

$$dp = \frac{-(n-1)dx}{nx^n} = qdt = nqx^{n-1}dx\ ;$$

$$\text{unde fit} \quad q = \frac{-(n-1)}{nnx^{2n-1}}\ ; \quad \&$$

$$dq = \frac{(n-1)(2n-1)dx}{nnx^{2n}} = rdt = nrx^{n-1}dx\,.$$

Hinc porro fit

$$r = \frac{(n-1)\ (2n-1)}{n^3x^{3n-1}}\ ; \ \& \ s = -\frac{(n-1)(2n-1)(3n-1)}{n^4x^{4n-1}}\,.$$

Quare ſi differentiale ipſius x^n ponatur conſtans, erit:

$$ddx = -\frac{(n-1)\,dx^2}{x}$$

$$d^3x = \frac{(n-1)\,(2n-1)\,dx^3}{xx}$$

$$d^4x = -\frac{(n-1)\,(2n-1)\,(3n-1)\,dx^4}{x^3}$$

&c.

266. Si expreſſio duas contineat variabiles x & y; earumque unius x differentiale poſitum ſit conſtans, ob $ddx = 0$, alia differentialia ſecunda & altiora non inerunt, praeter ddy, d^3y, &c. Haec autem eodem modo, quo ante uſi ſumus, tolli poterunt ponendo

$$dy = pdx;\; dp = qdx;\; dq = rdx;\; dr = sdx;\; \&c.$$

fiet enim $ddy = qdx^2;\; d^3y = rdx^3;\; d^4y = sdx^4$ &c.
quibus ſubſtitutis expreſſio orietur, quae praeter quantitates finitas x, y, p, q, r, s, &c. nonniſi differentiale primum dx continebit. Sic ſi propoſita fuerit haec expreſſio

$$\frac{ydx^4 + xdyd^3y + xd^4y}{(xx + yy)\,ddy}$$

in qua dx eſt conſtans aſſumtum; ponatur

$$dy = pdx;\; dp = qdx;\; dq = rdx;\; \&\; dr = sdx;$$

quibus valoribus ſubſtitutis expreſſio propoſita tranſmutabitur in hanc: $\frac{(y + xpr + xs)\,dx^2}{(xx + yy)q}$ quae nulla amplius differentialia ſecunda altiorave continet.

267. Simili modo differentialia ſecunda & altiora tollentur, ſi dy fuerit conſtans aſſumtum. Verum ſi aliud differentiale primum quodcunque dt ſtatuatur conſtans, tum primum modo ante indicato differentialia ipſius x altiora ex calculo tollantur, ponendo

$$dx = pdt;\; dp = qdt;\; dq = rdt;\; dr = sdt;\; \&c.$$

unde fit

$$ddx = qdt^2;\; d^3x = rdt^3;\; d^4x = sdt^4;\; \&c.$$

Deinde ſimili modo differentialia altiora ipſius y ponendo

$$dy = \mathrm{P}dt;\; d\mathrm{P} = \mathrm{Q}dt;\; d\mathrm{Q} = \mathrm{R}dt;\; d\mathrm{R} = \mathrm{S}dt;\; \&c.$$

unde fiet

$$ddy = \mathrm{Q}dt^2;\; d^3y = \mathrm{R}dt^3;\; d^4y = \mathrm{S}dt^4;\; \&c.$$

quibus ſubſtitutis obtinebitur expreſſio, quae praeter quantitates finitas, x, p, q, r, s, &c. y, P, Q, R, S, &c. ſolum differentiale dt complectetur, neque propterea vagam habebit ſignificationem.

268. Si differentiale primum, quod conſtans ponitur, vel ab x vel ab y vel ab utroque ſimul pendet, tum non opus

opus eſt, ut duplex quantitatum finitarum p, q, r, &c. ſeries introducatur. Si enim dx ab x tantum pendet, tum litterae p, q, r, &c. fient functiones ipſius x, ſolaeque litterae P, Q, R, &c. ingrediuntur; idemque evenit, ſi differentiale conſtans dx ab y tantum pendeat. At ſi dx ab utraque pendeat, operatio aliquantum immutari debet. Ponamus exempli gratia hoc differentiale ydx conſtans eſſe aſſumtum, eritque $yddx + dxdy = 0$; unde fit $ddx = -\frac{dxdy}{y}$. Sit nunc $dy = pdx$; $dp = qdx$; $dq = rdx$ &c. eritque $ddx = -\frac{pdx^2}{y}$; ulteriuſque differentiando

$$d^3x = -\frac{qdx^3}{y} + \frac{ppdx^3}{yy} - \frac{2pdxddx}{y},$$

ſubſtituatur hic loco ddx eius valor $-\frac{pdx^2}{y}$; fiet

$$d^3x = -\frac{qdx^3}{y} + \frac{3ppdx^3}{yy}; \text{ porroque}$$

$$d^4x = -\frac{rdx^4}{y} + \frac{pqdx^4}{yy} + \frac{6pqdx^4}{yy} - \frac{6p^3dx^4}{y^3} + \left(\frac{3pp}{yy} - \frac{q}{y}\right) 3dx^2ddx;$$

& pro ddx ſubſtituto valore $-\frac{pdx^2}{y}$ emerget

$$d^4x = \left(\frac{-r}{y} + \frac{10pq}{yy} - \frac{15p^3}{y^3}\right) dx^4 \text{ \&c.}$$

Deinde cum ſit $dy = pdx$; erit

ddy

$$ddy = qdx^2 + pddx = \left(q - \frac{pp}{y}\right) dx^2;$$

& continuo, pro ddx valore $-\frac{pdx^2}{y}$ ſubſtituendo fiet

$$d^3y = \left(r - \frac{4pq}{y} + \frac{3p^3}{yy}\right) dx^3, \quad \&$$

$$d^4y = \left(s - \frac{7pr}{y} - \frac{4qq}{y} + \frac{25ppq}{yy} - \frac{15p^4}{y^3}\right) dx^4 \quad \&c.$$

qui valores loco differentialium altiorum ipſarum x & y ſubſtituti mutabunt expreſſionem propoſitam in eiuſmodi formam, quae nulla amplius differentialia altiora continebit, hincque conſideratione cuiuſpiam differentialis conſtantis exuetur. Facta enim hac transformatione, quia differentialia ſecunda non inſunt, nequidem opus eſt, ut quale differentiale ſumtum ſit conſtans, commemoretur.

269. Saepiſſime autem in calculo ad lineas curvas applicato evenire ſolet, ut hoc differentiale primum $\surd(dx^2+dy^2)$ conſtans aſſumatur: quare quemadmodum hoc caſu differentialia ſecunda & altiora eliminari debeant, oſtendamus. Sic enim ſimul via patebit ad idem negotium abſolvendum, ſi aliud quodcunque differentiale aſſumendum ſit conſtans. Ponatur iterum $dy = pdx$; $dp = qdx$; $dq = rdx$; $dr = sdx$; &c. atque differentiale $\surd(dx^2 + dy^2)$ induet hanc formam $dx\surd(1 + pp)$, quae cum ſit conſtans fiet

$$ddx\surd(1 + pp) + \frac{pqdx^2}{\surd(1 + pp)} = 0,$$

ideoque $$ddx = -\frac{pqdx^2}{1 + pp};$$

unde iam ipſius ddx valor habebitur: hinc porro erit

d^3x

$$d^3x = -\frac{prdx^3}{1+pp} - \frac{qqdx^3}{1+pp} + \frac{2ppqqdx^3}{(1+pp)^2} - \frac{2pqdxddx}{1+pp}$$

$$= -\frac{prdx^3}{1+pp} - \frac{qqdx^3}{1+pp} + \frac{4ppqqdx^3}{(1+pp)^2}$$

$$= -\frac{prdx^3}{1+pp} + \frac{(3pp-1)qqdx^3}{(1+pp)^2}.$$

Deinde fiet

$$d^4x = -\frac{psdx^4}{1+pp} + \frac{(10pp-3)qrdx^4}{(1+pp)^2} - \frac{(15pp-13)pq^3dx^4}{(1+pp)^3}.$$

Quia autem aſſumſimus $dy = pdx$, fiet differentiando

$$ddy = qdx^2 + pddx = qdx^2 - \frac{ppqdx^2}{1+pp} = \frac{qdx^2}{1+pp},$$

$$d^3y = \frac{rdx^3}{1+pp} - \frac{2pqqdx^3}{(1+pp)^2} + \frac{2qdxddx}{1+pp}, \quad \text{ideoque}$$

$$d^3y = \frac{rdx^3}{1+pp} - \frac{4pqqdx^3}{(1+pp)^2};$$

porroque differentiando:

$$d^4y = \frac{sdx^4}{1+pp} - \frac{13pqrdx^4}{(1+pp)^2} + \frac{4(6pp-1)q^3dx^4}{(1+pp)^3}.$$

Omnia ergo differentialia altiora utriuſque variabilis x & y per quantitates finitas & poteſtates ipſius dx exprimentur, atque poſt has ſubſtitutiones factas reſultabit expreſſio a differentialibus ſecundis prorſus libera.

270. Expoſito igitur modo differentialia ſecunda & altiora exuendi, conveniet hoc negotium aliquot exemplis illuſtrari.

I. Sit propoſita haec expreſſio $\frac{xddy}{dx^2}$, in qua dx poſitum

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eſt conſtans. Poſito ergo $dy = pdx$, & $dp = qdx$, ob $ddy = qdx^2$, expreſſio propoſita abit in hanc finitam xq.

II. Sit propoſita haec expreſſio $\frac{dx^2 + dy^2}{ddx}$, in qua poſitum ſit dy conſtans. Ponatur $dx = pdy$; $dp = qdy$, ob $ddx = qdy^2$, orietur $\frac{1+pp}{q}$. Sin autem ut ante ſtatuere velimus $dy = pdx$, $dp = qdx$; ob dy conſtans erit $0 = pddx + dpdx$ & $ddx = -\frac{qdx^2}{p}$; unde expreſſio propoſita tranſibit in $\frac{-p(1+pp)}{q}$.

III. Sit propoſita haec expreſſio $\frac{yddx - xddy}{dxdy}$ in qua ydx poſitum ſit conſtans. Ponatur $dy = pdx$ & $dp = qdx$, eritque ex §. 268: $ddx = -\frac{pdx^2}{y}$, $ddy = qdx^2 - \frac{ppdx^2}{y}$, quibus ſubſtitutis expreſſio propoſita tranſmutatur in hanc: $-1 - \frac{xq}{p} + \frac{xp}{y}$.

IV. Sit propoſita iſta expreſſio $\frac{dx^2 + dy^2}{ddy}$, in qua conſtans ſit poſitum $\sqrt{(dx^2 + dy^2)}$. Ponatur iterum $dy = pdx$, $dp = qdx$, & ex paragrapho praecedente erit $ddy = \frac{qdx^2}{1+pp}$; unde expreſſio propoſita abibit in $\frac{(1+pp)^2}{q}$.

Ex

Ex his autem exemplis ſatis intelligitur, quemadmodum in quovis caſu oblato, quodcunque differentiale primum aſſumtum ſit conſtans, differentialia ſecunda atque altiora eliminari debeant.

271. Cum igitur hoc modo introducendis quantitatibus finitis p, q, r, s, &c. differentialia ſecunda & altiora ita eliminari queant, ut tota expreſſio praeter quantitates finitas x, y, p, q, r, s, &c. ſolum differentiale dx complectatur; viciſſim ſi huiuſmodi expreſſio reducta proponatur, ea iterum in formam priorem tranſmutari poterit loco litterarum p, q, r, s, &c. introducendis differentialibus ſecundis & altioribus. Nunc autem perinde erit, quodnam differentiale primum conſtans aſſumatur; atque vel id ipſum, quod ante fuit aſſumtum conſtans poni poteſt, vel aliud quodcunque. Quin etiam prorſus nullum differentiale conſtans aſſumi poterit, hocque modo prodibunt expreſſiones differentialia ſecunda altiorave continentes, quae etiamſi nullum differentiale conſtans ſit aſſumtum, tamen fixas ſignificationes obtineant, cuiuſmodi expreſſiones dari ſupra oſtendimus.

272. Sit ergo propoſita expreſſio quaecunque continens litteras finitas x, y, p, q, r, &c. una cum differentiali dx, in qua ſit $p = \frac{dy}{dx}$; $q = \frac{dp}{dx}$; $r = \frac{dq}{dx}$; &c. Si enim has litteras p, q, r, &c. ita eliminare velimus, ut earum loco introducamus differentialia ſecunda & altiora ipſarum x & y, nullo differentiali conſtante aſſumto; fiet

$$dp = \frac{dxddy - dyddx}{dx^2}, \text{ hincque } q = \frac{dxddy - dyddx}{dx^3},$$

quae formula differentiata dabit

$$dq = \frac{dx^2 d^3 y - 3dxddxddy + 3dyddx^2 - dxdyd^3 x}{dx^4},$$

unde fit $$r = \frac{dx^2 d^3 y - 3dxddxddy + 3dyddx^2 - dxdyd^3 x}{dx^5}$$

 Quod

Quod ſi inſuper littera s, quae denotat valorem $\frac{dr}{dx}$, inſit, pro ea ſubſtitui debebit hic valor $s =$

$$\frac{dx^3d^4y-6dx^2ddxd^3y-4dx^2ddyd^3x+15dxddx^2ddy+10dxdyddxd^3x-15dyddx^3-dx^2dyd^4x}{dx^7}$$

His igitur valoribus loco quantitatum p, q, r, s &c. ſubſtitutis expreſſio propoſita transmutabitur in aliam differentialia altiora ipſarum x & y continentem, quae etiamſi nullum differentiale primum conſtans ſit aſſumtum, tamen non vagam ſed fixam habebit ſignificationem.

273. Hoc ergo modo quaevis formula differentialis altioris gradus, in qua quodpiam differentiale primum aſſumtum eſt conſtans, tranſmutari poterit in aliam formam, in qua nullum differentiale conſtans ponitur, quae hoc non obſtante eundem valorem fixum habeat. Primum ſcilicet ope methodi ante traditae aſſumtis valoribus

$$dy = pdx;\ dp = qdx;\ dq = rdx;\ dr = sdx;\ \&c.$$

differentialia altiora eliminentur, tum loco p, q, r, s, &c. valores nunc inventi ſubſtituantur, atque orietur expreſſio priori aequalis nullum differentiale conſtans involvens: quam transformationem exempla ſequentia illuſtrabunt.

I. Sit propoſita haec expreſſio $\frac{xddy}{dx^2}$, in qua dx poſitum conſtans, quae tranſmutari debeat in aliam formam nullum differentiale conſtans involventem. Ponatur $dy = pdx$; $dp = qdx$; atque ut ante (270) vidimus expreſſio propoſita tranſibit in hanc: qx. Nunc loco q ſubſtituatur valor, quem obtinet nullo differentiali conſtanti aſſumto $q = \frac{dxddy - dyddx}{dx^3}$ atque reperietur haec expreſſio $\frac{xdxddy - xdyddx}{dx^3}$ propoſitae aequalis, & nullum amplius differentiale conſtans involvens.

II.

II. Sit propoſita haec expreſſio $\frac{dx^2 + dy^2}{ddx}$; in qua dy aſſumtum eſt conſtans. Ponatur $dy = pdx$ & $dp = qdx$; eaque tranſibit in hanc: $-\frac{p(1+pp)}{q}$, ſtatuatur nunc $p = \frac{dy}{dx}$ & $q = \frac{dxddy - dyddx}{dx^3}$, atque invenietur: $\frac{dy(dx^2 + dy^2)}{dyddx - dxddy}$ quae nullo differentiali aſſumto conſtante eundem fixum habet valorem, quem propoſita.

III. Sit propoſita haec expreſſio: $\frac{yddx - xddy}{dxdy}$, in qua differentiale ydx conſtans eſt aſſumtum. Ponatur $dy = pdx$, atque uti ſupra (270) vidimus haec expreſſio tranſmutatur in hanc: $-1 - \frac{xq}{p} + \frac{xp}{y}$, quae nullo differentiali conſtante aſſumto transformabitur in iſtam:

$$-1 - \frac{xdxddy - xdyddx}{dx^2dy} + \frac{xdy}{ydx}$$

$$= \frac{xdxdy^2 - ydx^2dy - yxdxddy + yxdyddx}{ydx^2dy}.$$

IV. Sit propoſita haec expreſſio $\frac{dx^2 + dy^2}{ddy}$, in qua conſtans aſſumtum eſt differentiale $\sqrt{(dx^2 + dy^2)}$. Poſito $dy = pdx$, & $dp = qdx$, orietur haec expreſſio $\frac{(1+pp)^2}{q}$, (loco citato). Statuatur nunc $p = \frac{dy}{dx}$, & $q = \frac{dxddy - dyddx}{dx^3}$, atque nullo aſſumto differentiali con-

ſtan-

ſtante nanciſcemur iſtam expreſſionem $\frac{(dx^2+dy^2)^2}{dx^2ddy - dxdyddx}$ propoſitae aequivalentem.

V. Sit propoſita haec expreſſio $\frac{dxd^3y}{xddy}$, in qua differentiale dx conſtans ſit aſſumtum. Ponatur

$$dy = pdx;\quad dp = qdx \quad \& \quad dq = rdx;$$

atque ob

$$ddy = qdx^2 \quad \& \quad d^3y = rdx^3$$

formula propoſita abibit in hanc $\frac{rdx^2}{xq}$. Nunc loco q & r ſubſtituantur valores, quos nullo differentiali conſtante aſſumto recipiunt ſcilicet: $q = \frac{dxddy - dyddx}{dx^3}$, &

$$r = \frac{dx^2d^3y - 3dxddxddy + 3dyddx^2 - dxdyd^3x}{dx^5}$$

atque obtinebitur ſequens expreſſio propoſitae aequivalens:

$$\frac{dx^2d^3y - 3dxddxddy + 3dyddx^2 - dxdyd^3x}{x(dxddy - dyddx)}$$

$$= \frac{dx(dxd^3y - dyd^3x)}{x(dxddy - dyddx)} - \frac{3ddx}{x}.$$

274. Si has transformationes diligentius intueamur, methodum eas perficiendi colligere poterimus expeditiorem, ita ut non opus ſit litteras p, q, r, &c. introducere. Varii autem modi hoc opus abſolvendi occurrent, prout aliud atque aliud differentiale in formula propoſita conſtans fuerit aſſumtum. Ponamus primum in formula propoſita differentiale dx conſtans eſſe aſſumtum; & quia loco dy poſuimus pdx, rurſuſque $\frac{dy}{dx}$ loco p: differentialia prima dx & dy,

ubi-

ubicunque in expreſſione occurrunt, ſine alteratione relinquuntur. Ubi autem occurrit ddy, quia eius loco ſcribitur qdx^2, & porro loco q valor $\frac{dxddy - dyddx}{dx^3}$, tranſmutatio abſolvetur, ſi ubique loco ddy ſtatim ponatur $\frac{dxddy - dyddx}{dx}$ ſeu $ddy - \frac{dyddx}{dx}$. Si inſuper in expreſſione propoſita occurrat d^3y, quia eius loco ponitur rdx^3, ob valorem ipſius r ante inventum, ubique loco d^3y ſcribi debebit

$$d^3y - \frac{3ddxddy}{dx} + \frac{3dyddx^2}{dx^2} - \frac{dyd^3x}{dx},$$

quo facto expreſſio propoſita tranſmutabitur in aliam, quae nullum differentiale conſtans involvit. Sic ſi proponatur iſta expreſſio $\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dxddy}$, in qua dx poſitum eſt conſtans, ei aequalis erit poſito $ddy - \frac{dyddx}{dx}$ loco ddy, haec nullum differentiale conſtans involvens: $\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dxddy - dyddx}$.

275. Hinc facile colligitur, ſi in expreſſione quapiam propoſita aſſumtum fuerit differentiale dy conſtans, tum ubique loco ddx ſcribi debere $ddx - \frac{dxddy}{dy}$, & loco d^3x hoc $d^3x - \frac{3ddxddy}{dy} + \frac{3dxddy^2}{dy^2} - \frac{dxd^3y}{dy}$; ut obtineatur expreſſio aequivalens, in qua nullum differentiale conſtans ponatur. Sin autem in expreſſione propoſita conſtans fuerit aſſumtum ydx, quoniam ſit $ddx = -\frac{tdx^2}{y}$, &

ddy

$ddy = qdx^2 - \frac{ppdx^2}{y}$; loco ddx ubique ſcribi debebit $-\frac{dxdy}{y}$, & loco ddy ubique $ddy - \frac{dyddx}{dx} - \frac{dy^2}{y}$: ad altiora differentialia, quia in hoc negotio rariſſime occurrere ſolent, non progredior. Quod ſi vero in expreſſione propoſita hoc differentiale $\surd(dx^2 + dy^2)$ aſſumtum fuerit conſtans, quia invenimus $ddx = -\frac{pqdx^2}{1+pp}$ & $ddy = \frac{qdx^2}{1+pp}$: pro ddx ubique ſcribi debet $\frac{dy^2\, ddx - dxdyddy}{dx^2 + dy^2}$ & loco ddy ubique $\frac{dx^2 ddy - dxdyddx}{dx^2 + dy^2}$. Sic ſi propoſita fuerit expreſſio $\frac{dy\surd(dx^2 + dy^2)}{ddx}$, in qua $\surd(dx^2 + dy^2)$ aſſumtum ſit conſtans, ea tranſmutabitur in hanc: $\frac{(dx^2 + dy^2)^2}{dyddx - dxddy}$, in qua nullum differentiale conſtans aſſumitur.

276. Quo iſtae reductiones facilius ad uſum accommodari queant, eas in ſequenti tabella complecti viſum eſt.

Formula igitur differentialis altioris gradus in aliam nullum differentiale conſtans involventem tranſmutabitur ope ſubſtitutionum ſequentium:

I. Si differentiale dx fuerit conſtans aſſumtum

loco	ſcribatur
ddy	$ddy - \frac{dyddx}{dx}$
d^3y	$d^3y - \frac{3ddxddy}{dx} + \frac{3dyddx^2}{dx^2} - \frac{dyd^3x}{dx}$

II.

II. Si differentiale dy fuerit conſtans aſſumtum

loco	ſcribatur
ddx	$ddx - \frac{dxddy}{dy}$
d^3x	$d^3x - \frac{3ddxddy}{dy} + \frac{3dxddy^2}{dy^2} - \frac{dxd^3y}{dy}$

III. Si differentiale ydx fuerit conſtans aſſumtum.

loco	ſcribatur
ddx	$-\frac{dxdy}{y}$
ddy	$ddy - \frac{dyddx}{dx} - \frac{dy^2}{y}$
d^3x	$\frac{dyddx}{y} - \frac{dxddy}{y} + \frac{3dxdy^2}{yy}$
d^3y	$d^3y - \frac{3ddxddy}{dx} + \frac{3dyddx^2}{dx^2} - \frac{dyd^3x}{dx} - \frac{4dyddy}{y} + \frac{4dy^2ddx}{ydx} + \frac{3dy^3}{yy}.$

IV. Si differentiale $\sqrt{(dx^2 + dy^2)}$ fuerit conſtans aſſumtum.

loco	ſcribatur
ddx	$\frac{dy^2ddx - dxdyddy}{dx^2 + dy^2}$
ddy	$\frac{dx^2ddy - dxdyddx}{dx^2 + dy^2}$
d^3x	$\frac{dy^2d^3x - dxdyd^3y}{dx^2 + dy^2} + \frac{(dxddy - dyddx)(3dy^2ddy - dx^2ddy + 4dxdyddx)}{(dx^2 + dy^2)^2}$
d^3y	$\frac{dx^2d^3y - dxdyd^3x}{dx^2 + dy^2} + \frac{(dyddx - dxddy)(3dx^2ddx - dy^2ddx + 4dxdyddy)}{(dx^2 + dy^2)^2}$

277. Expreſſiones ergo iſtae, quae nullum differentiale conſtans includunt, ita erunt comparatae, ut pro lubitu quodvis differentiale conſtans aſſumi queat. Hincque expreſſiones differentiales altiorum graduum, in quibus uullum differentiale conſtans aſſumtum perhibetur, examinari poſſunt, utrum ſignificatio earum ſit vaga an fixa. Ponatur enim pro lubitu quodpiam differentiale puta dx conſtans, tum per regulam §. praeced. priorem reducatur expreſſio iterum ad formam, in qua nullum differentiale conſtans ſit aſſumtum, quae ſi cum propoſita conveniat, ea erit fixa, neque ab inconſtantia differentialium ſecundorum pendebit: ſin autem expreſſio prodeat diverſa, tum propoſita vagam habet ſignificationem. Sic ſi ponatur haec expreſſio $yddx - xddy$, in qua nullum differentiale poſitum ſit conſtans; ad inveſtigandum, utrum ſignificationem fixam habeat an vagam, ponatur dx conſtans, eaque abibit in $-xddy$; nunc per regulam primam §. praeced. loco ddy ponatur, $ddy - \frac{dyddx}{dx}$ ac prodibit $-xddy + \frac{xdyddx}{dx}$, cuius a propoſita diſcrepantia indicat, propoſitam expreſſionem fixam ſtatamque ſignificationem non habere.

278. Simili modo ſi proponatur expreſſio generalis huiuſmodi $Pddx + Qdxdy + Rddy$, conditio definiri poterit, ſub qua ea nullo differentiali conſtante aſſumto valorem fixum habeat. Ponatur enim dx conſtans, atque expreſſio propoſita abibit in hanc $Qdxdy + Rddy$: nunc haec iterum transformetur in aliam formam, ut eius ſignificatus idem maneat, etiamſi nullum differentiale conſtans fingatur, ſicque prodibit $Qdxdy + Rddy - \frac{Rdyddx}{dx}$, quae forma cum propoſita congruet, ſi fuerit $Pdx + Rdy = 0$; hocque ſolo caſu valor eius erit fixus. Verum ſi non fuerit $P = -\frac{Rdy}{dx}$ ſeu $R = -\frac{Pdx}{dy}$, tum expreſſio propoſita $Pddx + Qdxdy + Rddy$ valorem fixum

xum non habebit, ſed eius ſignificatio erit vaga atque diverſa, prout aliud atque aliud differentiale conſtans aſſumitur.

279. Ex his principiis etiam facile erit expreſſionem differentialem, in qua quodpiam differentiale conſtans eſt poſitum, tranſmutare in aliam formam, in qua aliud differentiale conſtans aſſumatur. Reducatur enim primum ad eiuſmodi formam, quae nullum differentiale conſtans involvat, quo facto illud alterum differentiale conſtans ponatur. Sic ſi in expreſſione propoſita differentiale dx aſſumtum ſit conſtans, eaque tranſmutanda ſit in aliam, quae differentiale dy conſtans implicet: in formulis ſupra loco ddy & d^3y ſubſtituendis ob dy conſtans ponatur $ddy = 0$, $d^3y = 0$, atque quaeſito ſatisfiet, ſi loco ddy ſubſtituatur $-\frac{dyddx}{dx}$ & $\frac{3dyddx^2}{dx^2} - \frac{dyddx}{dx}$ loco d^3y. Hoc modo iſta formula $-\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxddy}$, in qua dx poſitum eſt conſtans, tranſmutabitur in hanc $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dyddx}$, in qua dy ponitur conſtans.

280. Si contra formula, in qua dy conſtans eſt poſitum, tranſmutari debeat in aliam; in qua dx ſit conſtans, tum loco ddx ſubſtitui debet $-\frac{dxddy}{dy}$ & loco d^3x haec expreſſio $\frac{3dxddy^2}{dy^2} - \frac{dxd^3y}{dy}$. Simili modo ſi formula, in qua $\sqrt{(dx^2+dy^2)}$ poſitum eſt conſtans, tranſmutari debeat in aliam, in qua dx ſit conſtans, tum loco ddx ſcribatur $-\frac{dxdyddy}{dx^2+dy^2}$ & $\frac{dx^2ddy}{dx^2+dy^2}$ loco ddy. At ſi formula, qua dx conſtans eſt aſſumtum, tranſ-

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mutari debeat in aliam, in qua $\sqrt{(dx^2 + dy^2)}$ ſit conſtans, quia ob $dx^2 + dy^2$ conſtans ſit $dxddx + dyddy = 0$, & $ddx = -\frac{dyddy}{dx}$, hoc valore loco ddx aſſumto, pro ddy ſcribi debebit $ddy + \frac{dy^2 ddy}{dx^2} = \frac{(dx^2 + dy^2)ddy}{dx^2}$. Sic haec formula $-\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dxddy}$, in qua dx eſt conſtans, tranſmutabitur in aliam, in qua $\sqrt{(dx^2 + dy^2)}$ ponitur conſtans, quae erit $-\frac{dx\sqrt{(dx^2 + dy^2)}}{ddy}$.

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CAPUT IX.

DE AEQUATIONIBUS DIFFERENTIALIBUS

281.

In hoc Capite imprimis eſt propoſitum earum functionum ipſius x, quae non explicite, ſed implicite per aequationem, qua relatio functionis iſtius y ad x continetur, definiuntur, differentiationem explicare: quo facto naturam aequationum differentialium in genere perpendemus, & quemadmodum ex aequationibus finitis oriantur, oſtendemus. Cum enim in calculo integrali ſummum negotium conſiſtat in integratione aequationum differentialium, ſeu in inventione eiuſmodi aequationum finitarum, qua cum differentialibus conveniant; neceſſe eſt, ut hoc loco indolem ac proprietates aequationum differentialium, quae ex earum origine ſequuntur, diligentius ſcrutemur, ſicque viam ad calculum integralem praeparemus.

282. Ut igitur hoc negotium abſolvamus, ſit y functio eiuſmodi ipſius x, quae per hanc aequationem quadratam $yy+Py+Q=0$ definiatur. Cum ergo haec expreſſio $yy+Py+Q$ ſit $=0$, quicquid x ſignificet, nihilo quoque aequalis erit, ſi loco x ſcribatur $x+dx$, quo caſu y abit in $y+dy$. Facta autem hac ſubſtitutione, ſi a quantitate reſultante ſubtrahatur prior $yy+Py+Q$, remanebit eius differentiale, quod propterea quoque erit $=0$. Hinc patet ſi expreſſio quaecunque fuerit $=0$, eius etiam differentiale fore aequale 0; atque ſi duae quaecunque expreſſiones inter ſe fuerint aequales earum quoque differentialia fore aequalia. Cum igitur ſit

$$yy+Py+Q=0,$$ erit quoque
$$2ydy+Pdy+ydP+dQ=0$$

quia

quia vero P & Q ſunt functiones ipſius x, earum differentialia huiuſmodi formam hebebunt,

$$dP = pdx, \ \& \ dQ = qdx;$$

unde fiet

$$2ydy + Pdy + ypdx + qdx = 0$$

ex qua oritur $$\frac{dy}{dx} = \frac{-yp-q}{2y+P}.$$

283. Quemadmodum ergo aequatio finita $yy + Py + Q = 0$ exponit relationem inter y & x, ita aequatio differentialis exprimit relationem ſeu rationem, quam dy tenet ad dx. Quoniam vero eſt $\frac{dy}{dx} = \frac{-yp-q}{2y+P}$, haec ratio $dy:dx$ cognoſci non poteſt, niſi ipſa functio y ſit cognita: neque vero res aliter ſe habere poteſt; cum enim ex aequatione finita y geminum obtineat valorem, uterque ſuum peculiare habebit differentiale, & utriuſque differentiale reperietur, prouti hic vel ille valor in expreſſione $\frac{-yp-q}{2y+P}$ loco y ſubſtituatur. Simili modo functio y per aequationem cubicam definiatur, valor functionis $\frac{dy}{dx}$ erit triplex; triplici ſcilicet ipſius y valori reſpondens. Si in aequatione propoſita finita y quatuor pluresve habeat dimenſiones, neceſſe eſt ut $\frac{dy}{dx}$ totidem ſignificationes ſortiatur.

284. Interim tamen ipſa functio y ex aequatione eliminari poterit, cum duae habeantur aequationes y continentes, finita ſcilicet & differentialis: tum autem eius differentiale dy ad totidem dimenſiones aſſurget, quot ante habuerat y, ſicque iſta aequatio omnes diverſas rationes ipſius dy ad dx ſimul complectetur. Sumamus praecedens exemplum aequationis $yy + Py + Q = 0$, cuius differentialis eſt:

2y-

$$2ydy + Pdy + ydP + dQ = 0,$$

ex qua fit $y = \frac{-Pdy - dQ}{2dy + dP}$ qui valor loco y in priori aequatione ſubſtitutus dabit:

$$(4Q - PP)dy^2 + (4Q - PP)dPdy + QdP^2 - PdPdQ + dQ^2 = 0,$$

cuius radices ſunt:

$$dy = -\tfrac{1}{2}dP \pm \frac{(\tfrac{1}{2}PdP - dQ)}{\sqrt{(PP - 4Q)}}$$

quae ſunt bina differentialia binorum ipſius y valorum ex aequatione finita: $y = -\tfrac{1}{2}P \pm \tfrac{1}{2}\sqrt{(PP - 4Q)}$.

285. Invento valore ipſius dy per repetitam differentiationem reperietur valor ipſius ddy, porroque ipſorum d^3y, d^4y, &c. qui autem, cum determinati non ſint, niſi aliquod differentiale primum conſtans ſtatuatur; ponamus commoditatis ergo dx conſtans, atque ad hoc oſtendendum ſumamus hoc exemplum $y^3 + x^3 = 3axy$, unde per differentiationem oritur $3yydy + 3xxdx = 3axdy + 3aydx$, hincque $\frac{dy}{dx} = \frac{ay - xx}{yy - ax}$, ſumantur denuo differentialia poſito dx conſtante atque invenietur

$$\frac{ddy}{dx} = \frac{-ayydy - aaxdy + 2xxydy - 2xyydx + aaydx + axxdx}{(yy - ax)^2}$$

ſubſtituatur loco dy eius valor modo inventus $\frac{aydx - xxdx}{yy - ax}$, atque diviſione per dx facta habebitur

$$\frac{ddy}{dx^2} = \frac{(ay - xx)(2xxy - ayy - aax)}{(yy - ax)^3} + \frac{axx + aay - 2xyy}{(yy - ax)^2}$$

ſeu $$\frac{ddy}{dx^2} = \frac{6axxyy - 2x^4y - 2xy^4 - 2a^3xy}{(yy - ax)^3}$$

$$= -\frac{2a^3xy}{(yy - ax)^3}$$

cum

cum ex aequatione finita ſit $2x^4y + 2xy^4 = 6axxyy$: hocque modo ope aequationis finitae hi valores in innumeras formas tranſmutari poſſunt.

286. Aequatio etiam differentialis prima infinitis modis poteſt variari, dum cum aequatione finita permiſcetur. Sic cum exemplo praecedente inventa eſſet aequatio differentialis

$$yydy + xxdx = axdy + aydx,$$

ſi ea multiplicetur per y, orietur

$$y^3dy + xxydx = axydy + ayydx,$$

in qua ſi loco y^3 ſubſtituatur eius valor $3axy - x^3$ orietur haec aequatio nova

$$2axydy - x^3dy + xxydx = ayydx;$$

quae denuo per y multiplicata, poſtquam loco y^3 eius valor fuerit ſubſtitutus, praebebit

$$2axy^2dy - x^3ydy + xxyydx = 3aaxydx - ax^3dx.$$

Generaliter autem ſi P, Q, R denotent functiones quaſcunque ipſarum x & y, ſi aequatio differentialis multiplicetur per P, erit

$$Pyydy + Pxxdx = aPxdy + aPydx.$$

Tum cum ſit $x^3 + y^3 - 3axy = 0$ erit quoque

$$(x^3 + y^3 - 3axy)(Qdx + Rdy) = 0,$$

quae aequationes invicem additae dabunt aequationem differentialem generalem ex propoſita aequatione finita natam

$$Pyydy - aPxdy + Rx^3dy + Ry^3dy - 3aRxydy +$$
$$Pxxdx - aPydx + Qx^3dx + Qy^3dx - 3aQxydx = 0.$$

287. Poſſunt vero etiam per ipſam differentiationem infinitae aequationes differentiales ex eadem aequatione finita inveniri, dum ea, antequam differentietur, per quantitatem quamcunque aut multiplicatur aut dividitur. Sic ſi P fuerit functio quaecunque ipſarum x & y, ut ſit $dP = pdx + qdy$, ſi aequatio finita per P multiplicetur, atque tum demum differentietur, obtinebitur aequatio differentialis generalis, quae infinitas formas diverſas induet, prouti pro P aliae atque aliae

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functiones aſſumuntur. Tum vero multiplicitas adhuc in infinitum augebitur, ſi ad hanc aequationem differentialem inventam addatur ipſa aequatio finita per huiuſmodi formulam $Qdx + Rdy$ multiplicata, ubi pro Q & R functiones quaſcunque ipſarum x & y aſſumere licet. Quanquam autem in his omnibus aequationibus relatio inter dy & dx, quam differentiale functionis y aequatione finita per x determinatae ad dx tenet, comprehenditur; tamen plerumque multo latius patent, & differentiale ipſius y per alias aequationes finitas determinati exprimit; cuius rei ratio in calculo integrali potiſſimum explicabitur.

283. Non ſolum autem ex eadem aequatione finita innumerabiles aequationes differentiales deduci poſſunt, ſed etiam plures imo infinitae exhiberi poſſunt aequationes finitae, quae ad eaſdem aequationes differentiales deducantur. Sic hae duae aequationes $yy = ax + ab$ & $yy = ax$ omnino ſunt diverſae, dum in priori quaecunque quantitas conſtans in locum ipſius b collocatur. Interim tamen hae ambae aequationes differentiatae eandem dant aequationem differentialem $2ydy = adx$; quin etiam omnes aequationes in hac forma $yy = ax$ contentae, quicunque valor ipſi a tribuatur, in una aequatione differentiali, in qua a non inſit, comprehendi poſſunt. Dividatur enim aequatio illa per x ut ſit $\frac{yy}{x} = a$, haecque differentiata dabit $2xdy - ydx = 0$. Poſſunt quoque aequationes tranſcendentes & algebraicae ad eandem aequationem differentialem perduci, uti fit in iſtis aequationibus

$$yy - ax = 0 \quad \& \quad yy - ax = bbe^{\frac{x}{a}},$$

ſi enim utraque per $e^{\frac{x}{a}}$ dividatur, ut habeantur iſtae aequationes: $e^{-\frac{x}{a}}(yy - ax) = 0$ & $e^{-\frac{x}{a}}(yy - ax) = bb$, ex utriuſque differentiatione orietur eadem differentialis

$$2ydy - adx - \frac{yydx}{a} + xdx = o.$$

289. Ratio huius diverſitatis in hoc conſiſtit, quod quantitatis conſtantis differentiale ſit $= o$. Quodſi ergo aequatio finita ad eiuſmodi formam reducatur, ut quantitas quaepiam conſtans ſola adſit, neque per variabiles vel multiplicetur vel dividatur; tum per differentiationem eruetur aequatio, in qua illa quantitas conſtans prorſus non adſit. Hoc modo quaelibet quantitas conſtans, quae in aequationem finitam ingreditur, per differentiationem tolli poteſt. Sic ſi propoſita fuerit aequatio $x^3 + y^3 = 3axy$; ſi ea per xy dividatur ut habeatur $\frac{x^3 + y^3}{xy} = 3a$, haec aequatio differentiata dabit:

$$2x^3ydx + 2xy^3dy - x^4dy - y^4dx = o,$$

quam conſtans a amplius non ingreditur.

290. Si plures quantitates conſtantes, quae in aequatione finita inſunt, tolli debeant, id fiet per differentiationem bis pluriesve repetitam; ſicque tandem obtinebuntur aequationes differentiales altiorum graduum iis conſtantibus prorſus carentes. Sit propoſita haec aequatio $yy = maa - nxx$, ex qua per differentiationem conſtantes maa & n tolli debeant. Prima quidem tolletur prima differentiatione, unde fit $ydy + nxdx = o$, hinc porro formetur aequatio $\frac{ydy}{xdx} + n = o$, quae ſumto dx conſtante, per differentiationem dabit:

$$xyddy + xdy^2 - ydxdy = o,$$

quae etſi nullam conſtantem complectitur, tamen omnes aequationes in hac forma $yy = maa - nxx$ contentas, quicunque valores litteris m, n & aa tribuantur, in ſe aeque comprehendit.

291. Non ſolum vero quantitates conſtantes, quae in aequationem finitam ingrediuntur, per differentiationem tolli poſſunt, ſed etiam altera variabilis, eius ſcilicet, cuius differentiale conſtans aſſumitur, per differentiationem eliminari po-

poterit. Ex aequatione enim inter x & y propoſita quaeratur valor x, ut ſit $x = Y$ denotante Y functionem ipſius y; eritque $dx = dY$, & ſumto dx conſtante, fiet differentiando $o = ddY$. Sin autem fuerit $xx + ax + b = Y$, fiet ter differentiando $o = d^3Y$, & aequatio $x^3 + axx + bx + c = Y$ quater differentiata dat $o = d^4Y$. Quanquam autem in his aequationibus una tantum variabilis ineſſe videtur, quae propterea variabilis eſſe ceſſaret, dum unica variabilis in nulla aequatione adeſſe poteſt; tamen quia differentiale dx conſtans eſt aſſumtum, eiuſque ratio in aequatione haberi debet, revera in aequationem ingredi cenſendum eſt. Hinc mirandum non eſt, ſi ſaepius aequationes differentiales ſecundi altiorisve gradus occurrant, in quibus unica tantum variabilis ineſſe videatur.

292. Praecipue autem notandum eſt, per differentiationem quantitates irrationales ac tranſcendentes ex aequatione tolli poſſe. Quod quidem ad irrationales attinet, quoniam per reductiones cognitas irrationalitas eliminari poteſt, hoc facto, per differentiationem aequatio obtinetur ab irrationalitate libera. Verum hoc ſaepenumero commodius ſine iſta reductione fieri poteſt, dum per comparationem aequationis differentialis cum finita formula irrationalis, ſi una tantum inſit, eliminari poteſt. Sin autem duae pluresve partes irrationales in aequatione finita contineantur, tum eius aequatio differentialis denuo differentietur, ſicque aequationes differentiales altiorum graduum tot quaerantur, quot requiruntur ad ſingulas partes irrationales eliminandas. Hoc modo etiam exponentes indefiniti pariter atque fracti tolli poterunt. Uti ſi fuerit $y^m = (aa - xx)^n$, poſt differentiationem habebitur

$$my^{m-1}\,dy = -2n(aa - xx)^{n-1}\,xdx,$$

quae per finitam diviſa dat $\frac{mdy}{y} = -\frac{2nxdx}{aa - xx}$, in qua nullus amplius exponens indefinitus occurrit. Hinc ergo patet aequationem differentialem ab omni irrationalitate liberam

ortam eſſe poſſe ex aequatione finita irrationali, atque adeo quantitates tranſcendentes involvente.

293. Ut autem intelligatur, quomodo per differentiationem quantitates tranſcendentes eliminentur, incipiamus a logarithmis, quorum differentialia cum ſint algebraica, negotium ſine difficultate abſolvetur. Sit enim $y = xlx$: erit $\frac{y}{x} = lx$, unde differentiando fit $\frac{xdy - ydx}{xx} = \frac{dx}{x}$, ideoque $xdy - ydx = xdx$. Si bini inſint logarithmi duplici differentiatione erit opus: ſit enim $ylx = xly$; erit $\frac{ylx}{x} = ly$, & differentiando, $\frac{xdylx + ydx - ydxlx}{xx} = \frac{dy}{y}$, ex qua concluditur fore $lx = \frac{xxdy - yydx}{yxdy - yydx}$. Haec aequatio iam iterum differentietur poſito dx conſtante, atque prodibit

$$\frac{dx}{x} = \frac{xxddy + 2xdxdy - 2ydxdy}{yxdy - yydx}$$

$$+ \frac{(yydx - xxdy)(yxddy + xdy^2 - ydxdy)}{(yxdy - yydx)^2} \text{ ſeu } \frac{dx}{x} =$$

$$\frac{y^3xdxddy - yyxxdxddy + 3yxxdxdy^2 - y^2xdxdy^2 + y^3dx^2dy - 2xyydx^2dy - x^3dy^3}{(yxdy - yydx)^2}$$

quae reducta dabit:

$$y^3xdxddy - yyxxdxddy + 3yxxdxdy^2 - 2xyydxdy^2$$
$$+ 3y^3dx^2dy - 2xyydx^2dy - x^3dy^3 - \frac{y^4dx^3}{x} = 0$$

ſeu $$yyxx(y - x)dxddy + 3yxdxdy(xxdy + yydx)$$
$$- 2yyxxdxdy(dx + dy) = x^4dy^3 + y^4dx^3.$$

194. Quantitates exponentiales ex aequatione eodem modo, quo logarithmi per differentiationem tolluntur. Si enim huiuſmodi propoſita fuerit $P = e^Q$, ubi P & Q fun-

ctio-

ctiones quaſcunque ipſarum x & y denotent; ea aequatio tranſmutari poterit in hanc logarithmicam $lP=Q$; cuius differentialis eſt $\frac{dP}{P}=dQ$ ſeu $dP=PdQ$. Neque obſtat, ſi quantitates exponentiales magis fuerint complicatae, tum enim ſi una differentiatio non ſufficit, duabus pluribuſve negotium abſolvetur.

I. Sit $y=\frac{e^x+e^{-x}}{e^x-e^{-x}}$; multiplicetur huius fractionis numerator ac denominator per e^x eritque $y=\frac{e^{2x}+1}{e^{2x}-1}$ unde fit

$$e^{2x}=\frac{y+1}{y-1} \quad \& \quad 2x=l\frac{y+1}{y-1},$$

cuius differentiale eſt $dx=-\frac{dy}{yy-1}=\frac{dy}{1-yy}$.

II. Sit $y=l\frac{e^x+e^{-x}}{2}$, fiet per primam differentiationem $dy=\frac{(e^x-e^{-x})dx}{e^x+e^{-x}}$, ſeu $\frac{dy}{dx}=\frac{e^{2x}-1}{e^{2x}+1}$, atque

$$e^{2x}=\frac{dy+dx}{dx-dy}. \quad \text{Ergo} \quad 2x=l\frac{dy+dx}{dx-dy}.$$

Sumto ergo dx conſtante erit

$$dx=\frac{dxddy}{dx^2-dy^2} \quad \text{ſeu} \quad dx^2=ddy+dy^2.$$

295. Simili modo quantitates tranſcendentes a circulo pendentes ex aequatione ope differentiationis tollentur, uti ex his exemplis intelligetur.

I. Sit $y=aA\,\text{ſin}\,\frac{x}{a}$; erit $dy=\frac{adx}{\sqrt{(aa-xx)}}$.

II. Sit $y=a\,\text{coſ}\,\frac{y}{x}$; erit $\frac{y}{a}=\text{coſ}\frac{y}{x}$, &

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$\frac{dy}{a} = -\frac{xdy + ydx}{xx} \text{ſin} \frac{y}{x}$. At cum ſit

$\text{coſ} \frac{y}{x} = \frac{y}{a}$; erit $\text{ſin} \frac{y}{x} = \frac{\surd(aa - yy)}{a}$;

quo valore ſubſtituto habebitur

$$\frac{dy}{a} = \frac{(ydx - xdy)\surd(aa - yy)}{axx}$$

ſeu $\quad xxdy = (ydx - xdy)\surd(aa - yy)$

III. Sit $y = m \text{ ſin } x + n \text{ coſ } x$, erit poſt differentiationem primam $dy = mdx \text{ coſ } x - ndx \text{ ſin } x$: quae denuo differentiata poſito dx conſtante dabit

$$ddy = -mdx^2 \text{ ſin } x - ndx^2 \text{ coſ } x,$$

haec autem per primam diviſa dat

$$\frac{ddy}{y} = -dx^2 \quad \text{ſeu} \quad ddy + ydx^2 = 0,$$

ex qua non ſolum ſinus & coſinus, ſed etiam conſtantes m & n evanuerunt.

IV. Sit $y = \text{ſin } lx$; erit $A \text{ ſin } y = lx$, unde per differentiationem fit $\frac{dy}{\surd(1 - yy)} = \frac{dx}{x}$; quae ſumtis quadratis dat $xxdy^2 = dx^2 - yydx^2$, haecque poſito dx conſtante ulterius differentiata praebet,

$$2xxdyddy + 2xdxdy^2 = -2ydx^2dy$$

ſeu $\quad xxddy + xdxdy + ydx^2 = 0.$

V. Sit $y = ae^{mx} \text{ ſin } nx$, erit differentiando

$$dy = mae^{mx}dx \text{ ſin } nx + nae^{mx}dx \text{ coſ } nx,$$

quae per propoſitam diviſa dat

$$\frac{dy}{y} = mdx + \frac{ndx \text{ coſ } nx}{\text{ſin } nx} = mdx + ndx \cot nx.$$

Erit ergo $A \cot\left(\frac{dy}{nydx} - \frac{m}{n}\right) = nx$. Quae aequatio poſito dx conſtante differentiata dat:

$$ndx = \frac{ndxdy^2 - nydxddy}{mmyydx^2 + nnyydx^2 - 2mydxdy}$$

ſeu $(mm + nn)yydx^2 - 2mydxdy = dy^2 - yddy$.
Perſpicuum igitur eſt, etiamſi in aequatione differentiali nullae quantitates tranſcendentes inſint, eam tamen ex aequatione finita oriri potuiſſe, quae a quantitatibus tranſcendentibus utcunque ſit affecta.

296. Quoniam igitur aequationes differentiales ſive primi ſive altioris gradus, quae duas variabiles x & y continent, ex aequationibus finitis oriuntur; iis etiam relatio inter binas iſtas variabiles exprimitur. Propoſita ſcilicet aequatione differentiali quacunque binas variabiles x & y complectente, ea ſignificatur certa quaedam relatio inter x & y, qua y ſit functio quaedam ipſius x. Hinc natura aequationis differentialis perſpicitur, ſi loco y ea ipſius x functio aſſignari poterit, quae per aequationem illam indicatur; ſeu quae ſit ita comparata, ut ſi ea ubique loco y, eiuſque differentiale loco dy, atque eius altiora differentialia loco ddy, d^3y, &c. ſubſtituantur, aequatio reſultet identica. In huius autem functionis inveſtigatione verſatur calculus integralis, cuius finis eo tendit, ut propoſita aequatione differentiali quacunque, functio illa ipſius x, cui altera variabilis y eſt aequalis, definiatur; ſeu quod eodem redit, ut aequatio finita inveniatur, qua relatio inter x & y contineatur.

297. Si exempli gratia proponatur aequatio haec

$$2ydy - adx - \frac{yydx}{a} + xdx = 0$$

ad quam ſupra §. 288 pervenimus, eiuſmodi relatio inter x & y ea definitur, quae ſimul hac aequatione finita

$yy - ax = bbe^{\frac{x}{a}}$ continetur. Cum igitur hinc ſit

$yy = ax + bbe^{\frac{x}{a}}$, patet $\sqrt{(ax + bbe^{\frac{x}{a}})} = y$
eam

eam eſſe functionem ipſius x, cui variabilis y vi propoſitae aequationis differentialis ſit aequalis. Namque ſi in aequatione loco yy, hunc valorem $ax + bbe^{\frac{x}{a}}$ & loco $2ydy$ eius differentiale $adx + \frac{bb}{a}e^{\frac{x}{a}}dx$ ſubſtituamus, orietur aequatio identica:

$$adx + \frac{bb}{a}e^{\frac{x}{a}}dx - adx - xdx - \frac{bb}{a}e^{\frac{x}{a}}dx + xdx = 0.$$

Sicque patet omnem aequationem differentialem aeque ac finitam certam relationem inter variabiles x & y exhibere, quae autem ſine ſubſidio calculi integralis reperiri nequeat.

298. Quo haec facilius intelligantur, ponamus cognitam eſſe eam functionem ipſius x, quae ipſi y vi cuiuſcunque aequationis differentialis ſive primi ſive altioris gradus, conveniat; ſitque

$$dy = pdx;\ dp = qdx;\ dq = rdx;\ \&c.$$

atque ſi in aequatione differentiale dx aſſumtum ſit conſtans, erit $ddy = qdx^2$, $d^3y = rdx^3$, &c. qui valores poſtquam in aequatione erunt ſubſtituti, ob omnes eius terminos homogeneos, differentialia dx per diviſionem evaneſcent, orieturque aequatio finitas tantum quantitates x, y, p, q, r, &c. complectens. Cum igitur ſint p, q, r, &c. quantitates a natura functionis y pendentes, aequatio revera tantum inter duas variabiles x & y ſubſiſtet; ſicque viciſſim conſtat, omni aequatione differentiali certam quandam relationem inter variabiles x & y determinari. Quamobrem ſi in ſolutione cuiuſvis problematis ad aequationem differentialem inter x & y perveniatur, per eam aeque relatio inter x & y exprimi cenſenda eſt, ac ſi ad aequationem finitam eſſet perventum.

299. Hoc igitur modo aequatio quaevis differentialis ita ad formam finitam reduci poteſt, ut in ea nonniſi quanti-

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tates finitae contineantur, differentialia autem ſeu infinite parva prorſus excedant. Cum enim ſit y certa functio ipſius x, ſi ponatur

$$dy=pdx;\ dp=qdx;\ dq=rdx;\ \&c.$$

quodcunque differentiale fuerit conſtans acceptum, differentialia ſecunda & altiora per poteſtates ipſius dx exprimentur, quae deinceps per diviſionem penitus tollentur. Ut ſi proponeretur haec aequatio

$$xyd^3y+xxdyddy+yydxddy-xydx^3=0$$

in qua dx ponitur conſtans; facto

$$dy=pdx,\ dp=qdx,\ \&\ dq=rdx,$$

ea abibit in

$$xyr+xxpq+yyq-xy=0,$$

poſtquam ſcilicet tota aequatio per dx^3 eſt diviſa. Haecque aequatio finita relationem inter x & y determinat.

300. Omnes ergo aequationes differentiales, cuiuſcunque ſint ordinis, his ſubſtitutionibus

$$dy=pdx;\ dp=qdx;\ dq=rdx;\ \&c.$$

ad meras quantitates finitas reducuntur. Atque ſi aequatio differentialis fuerit primi ordinis, ita ut differentialia prima eam tantum ingrediantur, per iſtam reductionem praeter variabiles y & x inſuper quantitas p introducetur. Sin autem aequatio differentialis fuerit ſecundi ordinis continens differentialia ſecunda, praeterea quantitas q; ac, ſi fuerit differentialis tertii ordinis, introducetur inſuper quantitas r, ſicque porro. Quoniam igitur hoc modo differentialia prorſus ex calculo exterminantur, ratio illa differentialis conſtantis penitus ceſſat; neque amplius, etiamſi inſint quantitates q, r, ex differentialibus ſecundis oriundae, opus erit indicare, an quodpiam differentiale conſtans ſit aſſumtum. Perinde enim eſt, utrum in evolutione aliquod differentiale pro lubitu conſtans ſtatuatur, an nullum.

301. Si igitur aequatio differentialis ſecundi vel altioris gradus proponatur, in qua nullum differentiale primum

conſtans eſſe aſſumtum perhibetur, hoc modo ſtatim explorari poterit, utrum ea determinatam relationem inter variabiles x & y contineat, nec ne. Quia enim nullum differentiale conſtans aſſumitur, in arbitrio noſtro relinquitur, quodnam differentiale conſtans ponere velimus; hincque tantum erit diſpiciendum, utrum diverſis differentialibus conſtantibus poſitis aequatio eandem relationem inter x & y exhibeat. Quodſi non eveniat, certum eſt ſignum, aequationem nullam determinatam relationem exprimere, ideoque in ſolutione nullius problematis locum habere poſſe. Tutiſſimus autem modus ſimulque facillimus hoc explorandi erit is ipſe, quem ſupra in ſimili negotio pro expreſſionibus differentialibus altiorum ordinum, num fixos habeant ſignificatus, dignoſcendis tradidimus.

302. Propoſita ergo huiuſmodi aequatione differentiali ſecundi altiorisve ordinis, in qua nullum differentiale conſtans ſit poſitum, ſtatuatur differentiale dx conſtans; deinde haec aequatio, uti ſupra de expreſſionibus differentialibus oſtendimus, iterum reducatur ad eiuſmodi formam, quae nullum differentiale conſtans ſupponat, ſtatuendo ſcilicet

$$ddy - \frac{dyddx}{dx} \text{ loco } ddy;$$

$$\& \quad d^3y - \frac{3ddxddy}{dx} + \frac{3dyddx^2}{dx^2} - \frac{dyd^3x}{dx} \text{ loco } d^3y,$$

&c.

Quo facto diſpiciatur, utrum aequatio hoc modo reſultans conveniat cum aequatione propoſita; quod ſi eveniat, aequatio propoſita determinatam relationem inter x & y complectetur; ſin autem ſecus accidat, aequatio erit vaga, neque definitam rationem inter variabiles x & y exprimet: quemadmodum hoc iam ante fuſius eſt demonſtratum.

303. Sit, quo hoc plenius explicetur, haec aequatio propoſita, quae nullo differentiali conſtante poſito reperta eſſe perhibeatur.

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$$Pddx + Qddy + Rdx^2 + Sdxdy + Tdy^2 = 0.$$

Ponatur dx conſtans, atque ea tranſibit in hanc:

$$Qddy + Rdx^2 + Sdxdy + Tdy^2 = 0.$$

Ex hac nunc iterum conſideratio differentialis conſtantis exuatur, modo ante praeſcripto, & obtinebitur:

$$-\frac{Qdyddx}{dx} + Qddy + Rdx^2 + Sdxdy + Tdy^2 = 0,$$

quae, quoniam a propoſita tantum ratione primi termini diſcrepat, videndum eſt, utrum ſit $P = -\frac{Qdy}{dx}$. Quod ſi deprehendatur, aequatio propoſita fixam relationem inter x & y exhibebit, quae per regulas in calculo integrali tradendas reperietur, quodcunque differentiale primum conſtans accipiatur. At, ſi fieri nequeat $P = -\frac{Qdy}{dx}$, aequatio propoſita erit impoſſibilis.

304. Niſi igitur haec propoſita aequatio:

$$Pddx + Qddy + Rdx^2 + Sdxdy + Tdy^2 = 0$$

ſit abſurda, neceſſe eſt ut ſit $Pdx + Qdy = 0$, quod duplici modo evenire poteſt: vel enim actu erit

$$P = -\frac{Qdy}{dx}, \text{ ſeu aequatio } Pdx + Qdy = 0$$

identica; vel erit $Pdx + Qdy = 0$ ipſa illa aequatio differentialis primi gradus, ex cuius differentiatione propoſita eſt orta: quo poſteriore caſu aequatio $Pdx + Qdy = 0$ congruet cum propoſita, eandemque relationem inter x & y continebit, ſicque ſine auxilio calculi integralis haec relatio erui poterit. Cum enim ſit $Pdx + Qdy = 0$, erit differentiando

$$Pddx + Qddy + dPdx + dQdy = 0,$$

quae ab aequatione propoſita ſubtracta relinquet:

$$Rdx^2 + Sdxdy + Tdy^2 = dPdx + dQdy.$$

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Cum autem ſit $dy = -\frac{Pdx}{Q}$, differentialia prorſus extingui poterunt, naſceturque aequatio finita inter x & y earum relationem indicans.

305. Ponamus in ſolutione problematis nullo differentiali conſtante aſſumto perventum eſſe ad hanc aequationem:

$$x^3 ddx + xxyddy - yydx^2 + xxdy^2 + aadx^2 = 0.$$

Erit ergo, cum aequationem abſurdum non continere conſtet: $x^3 dx + xxydy = 0$, ſeu $xdx + ydy = 0$: cuius differentiale erit

$$x^3 ddx + xxyddy + 3xxdx^2 + 2xydxdy + xxdy^2 = 0$$

quae aequatio a propoſita ſubtracta relinquit:

$$aadx^2 - yydx^2 - 3xxdx^2 - 2xydxdy = 0, \quad \text{ſeu}$$

$$aadx - yydx - 3xxdx - 2xydy = 0.$$

Cum autem ſit

$$xdx + ydy = 0; \quad \text{erit} \quad 2xydy = -2xxdx;$$

ideoque

$$aadx - yydx - xxdx = 0 \quad \text{ſeu} \quad yy + xx = aa;$$

quae aequatio veram relationem inter x & y exprimit, ſiquidem ea conſentit cum differentiali primum inventa $xdx + ydy = 0$. Qui conſenſus, niſi ſe manifeſtaſſet, aequatio propoſita pro impoſſibili eſſet habenda; cum autem hoc caſu locum habuerit, aequationem finitam $xx + yy = aa$ ſine calculo integrali elicere licuit.

306. Ut vero etiam exemplum aequationis impoſſibilis afferamus, propoſita ſit haec aequatio:

$$yyddx - xxddy + ydx^2 - xdy^2 + adxdy = 0,$$

in qua nullum differentiale conſtans ſit aſſumtum. Foret ergo $yydx - xxdy = 0$, ideoque differentiando

$$yyddx - xxddy + 2ydxdy - 2xdxdy = 0,$$

quae propoſitae aequalis poſita dabit:

$$ydx^2 - xdy^2 + adxdy = 2ydxdy - 2xdxdy.$$

Cum

Cum vero ſit $dy = \frac{yydx}{xx}$, extinguendis differentialibus obtinebitur: $y - \frac{y^4}{x^3} + \frac{ayy}{xx} = \frac{2y^3}{xx} - \frac{2yy}{x}$ ſeu

$$x^3 - y^3 + axy = 2xyy - 2xxy,$$

quae utrum cum differentiali $yydx - xxdy = 0$ conſentiat, eam differentiando, facile patebit, fiet enim:

$$3xxdx - 3yydy + axdy + aydx = 2yydx + 4xydy - 2xxdy - 4xydx$$

$$\text{ſeu } \frac{dy}{dx} = \frac{3xx + ay - 2yy + 4xy}{3yy - ax + 4xy - 2xx},$$

at ex illa eſt $\frac{dy}{dx} = \frac{yy}{xx}$, foretque ergo

$$3x^4 + 4x^3y + axxy = 3y^4 + 4xy^3 - axyy$$

$$\text{ſeu } axy = \frac{3y^4 + 4xy^3 - 4x^3y - 3x^4}{x + y}$$

$$= 3y^3 + xyy - xxy - 3x^3.$$

Verum ex aequatione finita primum inventa eſt

$$axy = y^3 + 2xyy - 2xxy - x^3,$$

quae ab iſta ſubtracta relinquit:

$0 = 2y^3 - xyy + xxy - 2x^3$, quae reſolvitur in has:

$$0 = y - x; \quad \& \quad 2yy + yx + 2xx = 0.$$

Quarum illa $y = x$ quidem cum differentiali $dy = \frac{yydx}{xx}$ conſtare poteſt, at vero aequationi finitae primum inventae adverſatur, niſi ſtatuatur $a = 0$, vel niſi utraque variabilis x & y conſtans ſtatuatur, quo quidem caſu ob $dx = 0$ & $dy = 0$ omnibus aequationibus differentialibus ſatisfit, aequatio propoſita ſubſiſtere nequit.

307. Conſideremus nunc etiam aequationes differentiales tres variabiles x, y, & z involventes, quae erunt vel primi, vel ſecundi, vel altioris gradus. Ad quarum naturam ſcrutandam no-

notari oportet, aequationem finitam tres variabiles comple-ctentem determinare relationem, quam unaquaeque ad binas reliquas teneat; definitur ergo, qualis functio ſit z ipſarum x & y. Quemadmodum igitur aequatio huiuſmodi finita reſolvitur, ſi reperiatur qualis functio ipſarum x & y loco z ſubſtitui debeat, ut aequationi ſatisfiat, ita quoque aequatio differentialis tres variabiles complectens determinabit, qualis functio una ſit reliquarum; iſque huiuſmodi aequationem reſolviſſe cenſendus eſt, qui indicaverit eam binarum variabilium x & y functionem, quae loco tertiae z ſubſtituta aequationi ſatisfaciat, ſeu eam identicam reddat. Aequatio ergo differentialis reſolvitur, ſi vel functio ipſarum x & y valorem ipſius z exhibens definiatur, vel aequatio finita aſſignetur, qua idem debitus ipſius z valor exprimatur.

308. Quanquam autem omnis aequatio differentialis duas tantum variabiles complectens, ſemper determinatam relationem inter eas exprimit; tamen hoc non ſemper evenit in aequationibus differentialibus trium variabilium. Dantur enim eiuſmodi aequationes, quibus plane nullo modo ſatisfieri poterit, quaecunque functio ipſarum x & y in locum ipſius z ſubſtituatur. Uti ſi propoſita fuerit haec aequatio $zdy = ydx$ facile patet, nullam prorſus dari functionem ipſarum x & y, quae loco z ſubſtituta reddat $zdy = ydx$, differentialia enim dx & dy nullo modo extinguentur. Simili modo apparet nullam dari functionem ipſarum x & z, quae loco y ſubſtituta eidem aequationi ſatisfaciat. Quaecunque enim pro y concipiatur functio ipſarum x & z, in eius differentiali dy ineſt dz, quod quia in aequatione non ineſt, deſtrui non poterit. Hancobrem nulla aequatio finita inter x, y, & z dari poteſt, quae aequationi differentiali $zdy = ydx$ conveniat.

309. Hinc aequationes differentiales tres variabiles continentes diſtribui oportet in imaginarias & reales. Huiuſmodi autem aequatio erit imaginaria ſeu abſurda, cui per nullam

aequa-

aequationem finitam ſatisfieri poteſt, cuiuſmodi erat illa $zdy = ydx$, quam modo conſideravimus. Aequatio autem erit realis, cui aequivalens aequatio finita exhiberi poteſt, quod evenit, ſi una variabilis aequalis ſit certae cuipiam functioni binarum reliquarum. Cuiuſmodi eſt haec aequatio:

$$zdy + ydz = xdz + zdx + xdy + ydx$$

congruit enim haec cum iſta aequatione finita:

$$yz = xz + xy \quad \text{ſitque} \quad z = \frac{xy}{y - x}.$$

Iſtud ergo diſcrimen inter huiuſmodi aequationes imaginarias & reales diligentiſſime eſt obſervandum; praecipue in calculo integrali, quia ridiculum foret, cuiuſpiam aequationis differentialis velle integralem, hoc eſt aequationem finitam ſatisfacientem quaerere, quae plane nullam habeat.

310. Primum igitur patet, omnes aequationes differentiales trium variabilium, in quibus tantum binarum differentialia occurrant, eſſe imaginarias & abſurdas. Ponamus enim in aequatione, quae contineat variabilem z, tantum ineſſe differentialia dx & dy, differentiale autem dz prorſus abeſſe; atque manifeſtum erit nullam exhiberi poſſe functionem ipſarum x & y, quae loco z ſubſtituta aequationem identicam producat; differentialia enim dx & dy nullo modo tolentur. His ergo caſibus omnino nulla datur aequatio finita ſatisfaciens: niſi forte eiuſmodi relatio inter x & y aſſignari queat, quae quicquid ſit z ſubſiſtere poſſit, uti fit in hac aequatione:

$$zdy - zdx = ydy - xdx,$$

cui ſatisfacit aequatio $y = x$. Facile autem inveſtigatur, quibus caſibus hoc eveniat, quaerendo relationem inter x & y primo ſi $z = 0$, & tum an iſta relatio aequationi pro quocunque ipſius z valore ſatisfaciat.

311. Neque vero ſolum aequatio tres variabiles involvens eſt abſurda, ſi duo tantum continet differentialia, ſed etiam ſi in ea omnia tria differentialia occurrant, talis eſſe

po-

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poterit. Quos caſus ut evolvamus, ponamus P & Q eſſe functiones ipſarum x & y tantum, atque haberi hanc aequationem

$$dz = Pdx + Qdy,$$

quae ſi non eſt abſurda, erit z functio quaepiam ipſarum x & y, cuius differentiale ſit

$dz = pdx + qdy$, eritque $P = p$ & $Q = q$.

At ſupra demonſtravimus $pdx + qdy$ non eſſe poſſe differentiale cuiuſquam functionis ipſarum x & y, niſi ſit $\left(\frac{dp}{dy}\right) = \left(\frac{dq}{dx}\right)$, denotante, uti ante aſſumſimus $\left(\frac{dp}{dy}\right)$ differentiale ipſius p poſita ſola y variabili, per dy diviſum, atque $\left(\frac{dq}{dx}\right)$ differentiale ipſius q, poſita ſola x variabili, diviſum per dx. Quocirca aequatio $dz = Pdx + Qdy$ realis eſſe nequit, niſi ſit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$.

312. Similis omnino erit ratio huius aequationis

$$dZ = Pdx + Qdy$$

ſi Z denotet functionem quamcunque ipſius z, P vero & Q ſint functiones ipſarum x & y, tertiam variabilem z non complectentes. Ut enim Z aequalis fieri poſſit functioni ipſarum x & y, neceſſe eſt ut ſit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$. Ex hoc ergo criterio aequatio differentialis quaeque propoſita, quae quidem in hac forma generali contineatur, diiudicari poteſt, utrum ſit realis an abſurda. Sic patebit hanc aequationem $zdz = ydx + xdy$ eſſe realem, nam ob

$P = y$ & $Q = x$, fit $\left(\frac{dP}{dy}\right) = 1 = \left(\frac{dQ}{dx}\right) = 1$.

Haec vero aequatio $azdz = yydx + xxdy$ eſt abſurda, ſit

fit enim $\left(\frac{dP}{dy}\right) = 2y$ & $\left(\frac{dQ}{dx}\right) = 2x$; qui valores ſunt inaequales.

313. Ut autem criterium latiſſime patens inveſtigemus, ſint P, Q, & R functiones quaecunque ipſarum x, y, & z; atque omnis aequatio differentialis trium variabilium, ſiquidem ſit primi gradus, continebitur in hac forma:

$$Pdx + Qdy + Rdz = 0.$$

Quoties ergo haec aequatio eſt realis, z aequabitur functioni cuipiam ipſarum x & y; eiuſque adeo differentiale erit huius formae $dz = pdx + qdy$. Quare ſi in aequatione propoſita iſta functio ipſarum x & y loco z, & $pdx + qdy$ loco dz ſubſtituatur, neceſſe eſt, ut prodeat aequatio identica $0 = 0$. Atque cum ex aequatione propoſita fiat: $dz = -\frac{Pdx}{R} - \frac{Qdy}{R}$, ſi in P, Q, & R valor ille loco z ſubſtituatur, neceſſe eſt ut fiat $p = -\frac{P}{R}$, & $q = -\frac{Q}{R}$.

314. Quoniam vero eſt $dz = pdx + qdy$, erit per ante demonſtrata $\left(\frac{dp}{dy}\right) = \left(\frac{dq}{dx}\right)$. Cum igitur ſubſtituto loco z ipſius valore in x & y fit

$$p = -\frac{P}{R} \quad \& \quad q = -\frac{Q}{R},$$

erit $$\left(\frac{dp}{dy}\right) = \left(-\frac{RdP + PdR}{RRdy}\right)$$

& $$\left(\frac{dq}{dx}\right) = \left(-\frac{RdQ + QdR}{RRdx}\right)$$

ideoque habebitur per RR multiplicando haec aequatio:

$$P\left(\frac{dR}{dy}\right) - R\left(\frac{dP}{dy}\right) = Q\left(\frac{dR}{dx}\right) - R\left(\frac{dQ}{dx}\right);$$

Ee ubi

ubi denominatores dy & dx iterum indicant, in differentialibus numeratorum eam ſolam quantitatem variabilem aſſumi debere, cuius differentiale denominatorem conſtituit. Haec autem differentialia dP, dQ, dR ante cognoſci non poſſunt, quam in ipſis quantitatibus P, Q, & R valor debitus loco z fuerit ſubſtitutus, qui autem cum ſit incognitus, ſequenti modo erit procedendum.

315. Quia P, Q, & R ſunt functiones ipſarum x, y, & z, ponamus

$$dP = \alpha dx + \beta dy + \gamma dz$$
$$dQ = \delta dx + \varepsilon dy + \zeta dz$$
$$dR = \eta dx + \theta dy + \iota dz$$

ubi α, β, γ, δ, ε, &c. denotant eas functiones, quae ex differentiatione oriuntur. Concipiamus nunc loco z ubique eius valorem in x & y expreſſum ſubſtitui, & loco dz, ponamus valorem $pdx + qdy$; fietque

$$dP = (\alpha + \gamma p)dx + (\beta + \gamma q)dy$$
$$dQ = (\delta + \zeta p)dx + (\varepsilon + \zeta q)dy$$
$$dR = (\eta + \iota p)dx + (\theta + \iota q)dy.$$

Ex his ergo valoribus erit:

$$\left(\frac{dR}{dy}\right) = \theta + \iota q \quad ; \quad \left(\frac{dR}{dx}\right) = \eta + \iota p$$
$$\left(\frac{dP}{dy}\right) = \beta + \gamma q \quad ; \quad \left(\frac{dQ}{dx}\right) = \delta + \zeta p.$$

316. Cum igitur ad realitatem aequationis requiratur, ut ſit:

$$P\left(\frac{dR}{dy}\right) - R\left(\frac{dP}{dy}\right) = Q\left(\frac{dR}{dx}\right) - R\left(\frac{dQ}{dx}\right),$$

fiet ſi inventi valores ſubſtituantur:

$$P(\theta + \iota q) - R(\beta + \gamma q) = Q(\eta + \iota p) - R(\delta + \zeta p).$$

At ante invenimus eſſe $p = -\frac{P}{R}$ & $q = -\frac{Q}{R}$

qui

qui valores, cum differentialia non amplius in computum veniant, adhiberi poterunt, etiamſi loco z eius valor in x & y non ſubſtituatur. Eritque ergo

$$P\theta - \frac{PQ\iota}{R} - R\beta + Q\gamma = Q\eta - \frac{PQ\iota}{R} - R\delta + P\zeta$$

ſeu $o = P(\zeta - \theta) + Q(\eta - \gamma) + R(\beta - \delta)$.
Quia autem quantitates β, δ, γ, η, ζ, θ, per differentiationem inveniuntur, erit ſuperiori notandi modo adhibito:

$$o = P\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + Q\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + R\left(\frac{dP}{dy} - \frac{dQ}{dx}\right).$$

Quae proprietas, niſi in aequatione locum habeat, aequatio non erit realis, ſed imaginaria & abſurda.

317. Quanquam hanc regulam ex conſideratione variabilis z elicuimus, tamen quia omnes quantitates aeque ingrediuntur, manifeſtum eſt, & reliquarum conſideratione, eandem expreſſionem prodituram fuiſſe. Propoſita ergo aequatione differentiali primi gradus, quae tres variabiles involvat, quacunque, ſtatim diiudicari poterit utrum ſit realis an imaginaria. Comparetur enim cum hac forma generali:

$$Pdx + Qdy + Rdz = o,$$

atque quaeratur valor huius formulae:

$$P\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + Q\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + R\left(\frac{dP}{dy} - \frac{dQ}{dx}\right),$$

qui ſi fuerit $= o$, aequatio erit realis, ſin autem non fuerit $= o$, certum hoc eſt ſignum, aequationem eſſe imaginariam ſeu abſurdam.

318. Aequatio propoſita per diviſionem quoque ſemper ad huiuſmodi formam reduci poteſt:

$$Pdx + Qdy + dz = o,$$

in quam, cum prior abeat ſi fiat $R = 1$, criterium ſimplicius exprimetur, hoc modo:

$$P\left(\frac{dQ}{dz}\right) - Q\left(\frac{dP}{dz}\right) + \left(\frac{dP}{dy}\right) - \left(\frac{dQ}{dx}\right) = o.$$

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Quoties enim haec expreſſio revera nihilo aequalis reperitur, toties aequatio propoſita erit realis; ſin autem contrarium eveniat, aequatio erit imaginaria. Poſterius quidem ex iis, quae demonſtravimus, eſt certum; de priori autem adhuc dubitari poſſit, utrum aequatio ſemper ſit realis, quoties quidem hoc criterium id indicat. Quod cum hoc loco pleniſſime demonſtrari nequeat, ſed in calculo demum integrali demonſtratione confirmari poſſit, hic tantum id affirmamus; neque autem periculum inde eſt metuendum, ſi quis tantiſper de eius veritate dubitare voluerit.

319. Ex hoc ergo criterio primum patet, ſi in aequatione $Pdx + Qdy + Rdz = 0$, fuerit P functio ipſius x, Q functio ipſius y, & R functio ipſius z tantum, aequationem ſemper fore realem. Fit enim

$$\left(\frac{dP}{dy}\right) = 0 \;;\; \left(\frac{dP}{dz}\right) = 0 \;;\; \left(\frac{dQ}{dz}\right) = 0 \;;$$

$$\left(\frac{dQ}{dx}\right) = 0 \;;\; \left(\frac{dR}{dx}\right) = 0 \;\&\; \left(\frac{dR}{dy}\right) = 0 \;;$$

ideoque tota expreſſio criterii ſponte evaneſcet.

320. Si fuerit ut ante P ipſius x, & Q ipſius y functio tantum, R autem functio quaecunque ipſarum x, y, & z, aequatio erit realis ſi fuerit:

$$P\left(\frac{dR}{dy}\right) = Q\left(\frac{dR}{dx}\right) \text{ ſeu } \left(\frac{dR}{dx}\right) : \left(\frac{dR}{dy}\right) = P:Q$$

Sic ſi propoſita fuerit haec aequatio:

$$\frac{2dx}{x} + \frac{3dy}{y} + \frac{x^2y^3dz}{z^6} = 0.$$

Quia hic eſt $P = \frac{2}{x}$; $Q = \frac{3}{y}$, & $R = \frac{x^2y^3}{z^6}$

hinc

hinc $\left(\frac{dR}{dx}\right) = \frac{2xy^3}{z^6}$; atque $\left(\frac{dR}{dy}\right) = \frac{3xxyy}{z^6}$;

erit $P\left(\frac{dR}{dy}\right) = Q\left(\frac{dR}{dx}\right) = \frac{6xyy}{z^6}$; ideoque aequatio propoſita erit realis.

321. Si fuerint P & Q functiones ipſarum x & y, at R functio ipſius z tantum, ob

$$\left(\frac{dP}{dz}\right) = 0; \left(\frac{dQ}{dz}\right) = 0; \left(\frac{dR}{dx}\right) = 0; \& \left(\frac{dR}{dy}\right) = 0,$$

aequatio erit realis ſi fuerit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$. Haec eadem vero conditio requiritur, ſi $Pdx + Qdy$ debeat eſſe differentiale determinatum, ſeu ex differentiatione cuiuſpiam functionis finitae ipſarum x & y ortum. Hucque redit quod ſupra §. 312 iam obſervavimus, aequationem $dZ = Pdx + Qdy$, ſi Z ſit functio ipſius z tantum at P & Q functiones ipſarum x & y, realem eſſe non poſſe, niſi ſit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$. Ambo autem iſti caſus inter ſe prorſus conveniunt: nam loco Rdz, ſi R eſt functio ipſius z tantum, poni poteſt dZ exiſtente Z functione ipſius z.

322. Ut hoc criterium inventum exemplo illuſtremus, conſideremus hanc aequationem:

$$(6xy^2z - 5yz^3)\,dx + (5x^2yz - 4xz^3)\,dy + (4x^2y^2 - 6xyz^2)dz = 0,$$

qua cum forma generali comparata fit:

$$P = 6xy^2z - 5yz^3; \quad \left(\frac{dP}{dy}\right) = 12xyz - 5z^3;$$

$$\left(\frac{dP}{dz}\right) = 6xy^2 - 15yz^2.$$

$$Q = 5x^2yz - 4xz^3; \quad \left(\frac{dQ}{dx}\right) = 10xyz - 4z^3;$$

$$\left(\frac{dQ}{dz}\right)=5x^2y-12xz^2;$$

$$R=4x^2y^2-6xyz^2;\quad \left(\frac{dR}{dx}\right)=8xy^2-6yz^2;$$

$$\left(\frac{dR}{dy}\right)=8x^2y-6xz^2.$$

His inventis valoribus aequatio iudicium continens erit haec:

$$+(6xy^2z-5yz^3)(-3xxy-6xzz)$$
$$+(5x^2yz-4xz^3)(2xyy+9yzz)$$
$$+(4x^2y^2-6xyz^2)(2xyz-z^3)=0.$$

Haec autem expreſſio ſi evolvatur, omnes termini actu ſe mutuo deſtruunt, fitque $0=0$, quod indicat aequationem propoſitam eſſe realem.

323. Quando autem expreſſio hoc modo ex criterio eruta non evaneſcit, tum id ſignum eſt aequationem propoſitam eſſe imaginariam. Quoniam vero hoc pacto ex criterio aequatio finita invenitur, ea, ſi quidem aequationi differentiali conveniat, ſimul relationem indicabit, quam variabiles inter ſe tenent. Atque hoc modo ii caſus, quorum ſupra meminimus (310), evolvuntur. Sit enim propoſita iſta aequatio:

$$(z-x)\,dx+(y-z)\,dy=0,$$

fiet $P=z-x$; $Q=y-z$; & $R=0$,

porro $\left(\frac{dP}{dz}\right)=1$, & $\left(\frac{dQ}{dz}\right)=-1$.

Aequatio iudicium exhibens fit $P\left(\frac{dQ}{dz}\right)=Q\left(\frac{dP}{dz}\right)$

ſeu $z-x=z-y$; unde fit $y=x$.

Quoniam igitur hic caſu evenit, ut aequatio $y=x$ ſimul aequationi differentiali ſatisfaciat, dicendum eſt propoſitam aequationem nil aliud ſignificare, niſi eſſe $y=x$.

324.

324. Propoſita ergo aequatione differentiali tres variabiles continente: $Pdx + Qdy + Rdz = 0$, tres conſiderandi erunt caſus ſequentes, ad quos haec aequatio deducit:

$$P\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + Q\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + R\left(\frac{dP}{dy} - \frac{dQ}{dx}\right) = 0.$$

Primus eſt ſi haec expreſſio revera ſit $= 0$, tumque aequatio propoſita erit realis. Sin autem haec aequatio finita non ſit identica, tum diſpiciendum eſt, utrum ea aequationi propoſitae ſatisfaciat: quodſi evenit, habebitur aequatio finita, qui eſt caſus ſecundus. Tertius autem caſus locum habet, ſi aequatio finita cum propoſita differentiali ſubſiſtere nequeat, atque tum aequatio propoſita erit imaginaria: neque enim ulla aequatio finita exhiberi poterit, quae ipſi ſatisfaciat.

325. Caſus primus ac tertius per ſe ſunt perſpicui, ſecundus autem, etſi rariſſime occurrit, probe tamen notari meretur: & cum eius exemplum iam ſupra in aequatione, quae duo tantum continet differentialia, exhibuerimus, etiam aequationem afferamus, in qua omnia tria differentialia inſint:

$(z - y)dx + xdy + (y - z)dz = 0$. Erit ergo:

$$P = z - y \;;\; \left(\frac{dQ}{dz}\right) = 0 \;;\; \left(\frac{dR}{dy}\right) = 1$$

$$Q = x \;;\; \left(\frac{dR}{dx}\right) = 0 \;;\; \left(\frac{dP}{dz}\right) = 1$$

$$R = y - z \;;\; \left(\frac{dP}{dy}\right) = -1 \;;\; \left(\frac{dQ}{dx}\right) = 1$$

unde aequatio finita criterium continens evadet:

$$z - x - y = 0, \text{ ſeu } z = x + y$$

ſubſtituatur hic valor pro z in aequatione differentiali fietque

$$xdx + xdy - x(dx + dy) = 0;$$

quae aequatio, cum ſit identica, ſequitur aequationem differentialem nil aliud ſignificare, niſi $z = x + y$.

326. Quoniam diximus omnes aequationes differentiales primi ordinis, in quibus tres variabiles inſunt contineri in hac forma $Pdx + Qdy + Rdz = 0$,

dubium hic naſci poterit circa eas aequationes, in quibus differentialia prima duas pluresve dimenſiones conſtituunt, cuiuſmodi eſt haec:

$$Pdx^2 + Qdy^2 + Rdz^2 = 2Sdxdy + 2Tdxdz + 2Vdydz.$$

Verum de huiuſmodi aequationibus notandum eſt, eas nullo modo reales eſſe poſſe, niſi habeant diviſores prioris formae, qui propterea aequationes ſimplices conſtituent. Cum enim ex hac aequatione fiat: $dz =$

$$\frac{Tdx + Vdy \pm \surd(dx^2(T^2 - PR) + 2dxdy(TV + RS) + dy^2(V^2 - QR))}{R}$$

facile patet z functioni cuipiam ipſarum x & y, ſeu dz huiuſmodi expreſſioni $pdx + qdy$ aequale fieri non poſſe, niſi quantitas irrationalis evadat rationalis, quod eveniet ſi fuerit:

$$(T^2 - PR)(V^2 - QR) = (TV + RS)^2$$

$$\text{ſeu} \quad R = \frac{PVV + 2STV + QTT}{PQ - SS}$$

Niſi ergo haec aequatio finita ipſi aequationi propoſitae ſatiſfaciat, haec erit imaginaria.

327. Supereſſet ut in hoc Capite quoque aequationes differentiales altiorum ordinum, quae tres variabiles complectuntur, perpenderemus, caſuſque definiremus, quibus eae vel reales vel imaginariae evadunt; verum quia criteria nimis fierent intricata, hunc laborem hic praetermittimus, praeſertim, cum ex iiſdem fontibus, quos hic aperuimus, ſequantur. Ceterum ſi in calculo integrali his criteriis erit opus, tum ea facile erui poterunt. Ob eandem cauſam hic quoque aequationes, quae plures variabiles complectuntur, non contemplamur, cum fere nunquam occurrant, atque, ſi unquam occurrerent, ex principiis hic traditis ſine negotio examinari poſſent. Quare his expoſitis Inſtitutioni Calculi Differentialis hic finem imponimus progreſſuri ad inſignes uſus oſtendendos, quos iſte calculus cum in ipſa Analyſi, tum in Geometria ſublimiori affert.

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INSTITUTIONUM

CALCULI DIFFERENTIALIS

PARS POSTERIOR

CONTINENS

USUM HUIUS CALCULI IN ANALYSI FINITORUM, NEC NON IN DOCTRINA SERIERUM.

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CAPUT I.

DE TRANSFORMATIONE SERIERUM.

I.

CUm nobis propoſitum ſit uſum Calculi differentialis tam in univerſa Analyſi, quam in doctrina de ſeriebus oſtendere; nonnulla ſubſidia ex Algebra communi, quae vulgo tractari non ſolent, hic erunt repetenda. Quae quamvis maximam partem iam in Introductione ſumus complexi, tamen quaedam ibi ſunt praetermiſſa, vel ſtudio quod expediat ea tum demum explicari, quando neceſſitas id exigat, vel quia cuncta, quibus opus ſit futurum, praevideri non poterant. Huc pertinet transformatio ſerierum, cui hoc Caput deſtinavimus, qua quaevis ſeries in innumerabiles alias ſeries tranſmutatur, quae omnes eandem habeant ſummam communem, ita ut, ſi ſeriei propoſitae ſumma ſit cognita, reliquae ſeries omnes ſimul ſummari queant. Hoc autem capite praemiſſo, eo uberius doctrinam ſerierum per calculum differentialem & integralem amplificare poterimus.

2. Conſiderabimus autem potiſſimum eiuſmodi ſeries, quarum ſinguli termini per poteſtates ſucceſſivas quantitatis cuiuſdam indeterminatae ſunt multiplicati: quoniam hae latius patent, maioremque utilitatem afferent.

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Sit igitur propoſita ſequens ſeries generalis, cuius ſummam, ſive ſit cognita ſive ſecus, ponamus $= S$, ſitque

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \&c.$$

Ponatur iam $x = \frac{y}{1+y}$, & cum ſit per ſeries infinitas

$$\begin{array}{l}
x = y - y^2 + y^3 - y^4 + y^5 - y^6 + \&c. \\
x^2 = y^2 - 2y^3 + 3y^4 - 4y^5 + 5y^6 - 6y^7 + \&c. \\
x^3 = y^3 - 3y^4 + 6y^5 - 10y^6 + 15y^7 - 21y^8 + \&c. \\
x^4 = y^4 - 4y^5 + 10y^6 - 20y^7 + 35y^8 - 56y^9 + \&c. \\
\&c.
\end{array}$$

hi valores ſubſtituti, ſerieque ſecundum poteſtates ipſius y diſpoſita, dabunt

$$\begin{array}{llllll}
S = ay & - ay^2 & + ay^3 & - ay^4 & + ay^5 & \&c. \\
 & + b & - 2b & + 3b & - 4b & \\
 & & + c & - 3c & + 6c & \\
 & & & + d & - 4d & \\
 & & & & + e. &
\end{array}$$

3. Quoniam poſuimus $x = \frac{y}{1+y}$; erit $y = \frac{x}{1-x}$, quo valore loco y ſubſtituto, ſeries propoſita

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \&c.$$

tranſmutabitur in hanc:

$$S = a.\frac{x}{1-x} + (b-a)\frac{x^2}{(1-x)^2} + (c-2b+a)\frac{x^3}{(1-x)^3} \&c.$$

in qua coefficiens ſecundi termini $b - a$ eſt differentia prima ipſius a ex ſerie $a, b, c, d, e,$ &c. quam ſupra per Δa expoſuimus; coefficiens tertii termini $c - 2b + a$ eſt differentia ſecunda $\Delta^2 a$; coefficiens quarti eſt differentia tertia $\Delta^3 a$, &c. Hinc differentiis ipſius a continuis, quae formantur ex ſerie $a, b, c, d, e,$ &c. adhibendis propoſita Series tranſmutabitur in hanc

$$S = \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a + \frac{x^3}{(1-x)^3}\Delta^2 a + \frac{x^4}{(1-x)^4}\Delta^3 a + \&c.$$

cuius ergo ſeriei ſumma habebitur, ſi propoſitae ſumma fuerit cognita.

4.

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4. Si igitur ſeries a, b, c, d, &c. ita fuerit comparata, ut tandem differentias habeat conſtantes, quod evenit, ſi eius terminus generalis fuerit functio rationalis integra, tum ſeries poſterior $\frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a +$ &c. tandem habebit terminos evaneſcentes, ſicque eius ſumma per expreſſionem finitam exhiberi poterit. Ita ſi ſeriei a, b, c, d, &c. differentiae primae iam fuerint conſtantes, tum ſeriei huius: $ax + bx^2 + cx^3 + dx^4 +$ &c. ſumma erit $= \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a$. At ſi illius ſeriei coefficientium differentiae ſecundae fiant conſtantes, tum ipſius ſeriei propoſitae erit

$$= \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a + \frac{x^3}{(1-x)^3}\Delta\Delta a.$$

Unde ſummae huiuſmodi ſerierum ex differentiis coefficientium facile invenientur.

I. *Quaeratur ſumma huius ſeriei:*

$$1x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \text{\&c.}$$

Diff. I. $\quad 2, \quad 2, \quad 2, \quad 2, \quad$ &c.

Cum ergo differentiae primae ſint conſtantes, ob $a = 1$ & $\Delta a = 2$, erit ſeriei propoſitae ſumma

$$= \frac{x}{1-x} + \frac{2xx}{(1-x)^2} = \frac{x+xx}{(1-x)^2}.$$

II. *Quaeratur ſumma huius ſeriei:*

$$1x + 4xx + 9x^3 + 16x^4 + 25x^5 + \text{\&c.}$$

Diff. I. $\quad 3, \quad 5, \quad 7, \quad 9, \quad$ &c.

Diff. II. $\quad 2, \quad 2, \quad 2, \quad$ &c.

Quia itaque eſt

$$a = 1;\ \Delta a = 3;\ \Delta^2 a = 2;$$

erit ſeriei propoſitae ſumma

$$= \frac{x}{1-x} + \frac{3xx}{(1-x)^2} + \frac{2x^3}{(1-x)^3} = \frac{x+xx}{(1-x)^3}.$$

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III. *Quaeratur ſumma huius ſeriei*

$$S = 4x + 15x^2 + 40x^3 + 85x^4 + 156x^5 + 259x^6 + \&c.$$

Diff. I. 11, 25, 45, 71, 103

Diff. II. 14, 20, 26, 32

Diff. III. 6, 6, 6

Quia eſt $a = 4$; $\Delta a = 11$; $\Delta^2 a = 14$; $\Delta^3 a = 6$; erit ſumma

$$S = \frac{4x}{1-x} + \frac{11xx}{(1-x)^2} + \frac{14x^3}{(1-x)^3} + \frac{6x^4}{(1-x)^4},$$

ſive $$S = \frac{4x - xx + 4x^3 - x^4}{(1-x)^4} = \frac{x(1+xx)(4-x)}{(1-x)^4}.$$

5. Quanquam hoc modo iſtarum ſerierum in infinitum progredientium ſummae inveniuntur; tamen ex iiſdem principiis hae Series quoque ad datum quemvis terminum ſummari poſſunt. Propoſita enim ſit haec ſeries

$$S = ax + bx^2 + cx^3 + dx^4 + \ldots\ldots + ox^n,$$

& quaeratur eius ſumma, ſi in infinitum progrediatur, quae erit $= \frac{x}{1-x}a + \frac{xx}{(1-x)^2}\Delta a + \frac{x^3}{(1-x)^3}\Delta^2 a + \&c.$ Nunc conſiderentur eiuſdem ſeriei termini poſt ultimum ox^n ſequentes, qui ſint

$$px^{n+1} + qx^{n+2} + rx^{n+3} + sx^{n+4} + \&c.$$

cuius ſeriei, ſi per x^n dividatur, ſumma, ut ante inveniri poterit; quae rurſus per x^n multiplicata erit

$$\frac{x^{n+1}}{1-x}p + \frac{x^{n+2}}{(1-x)^2}\Delta p + \frac{x^{n+3}}{(1-x)^3}\Delta^2 p + \&c.$$

quae ſumma ſi a totius ſeriei in infinitum continuatae ſumma ſubtrahatur, remanebit ſumma portionis propoſitae quaeſita: S =

$$\frac{x}{1-x}(a - x^n p) + \frac{x^2}{(1-x)^2}(\Delta a - x^n \Delta p) + \frac{x^3}{(1-x)^3}(\Delta^2 a - x^n \Delta^2 p)$$
&c.

I. *Quaeratur ſumma huius ſeriei finitae.*

$$S = 1x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots + nx^n.$$

Tam

Tam horum coefficientium, quam terminum ultimum ſequentium quaerantur differentiae:

$$\begin{array}{ccccc|ccc} 1, & 2, & 3, & 4, & \&c. & n+1, & n+2, & n+3, \&c. \\ & 1, & 1, & 1, & & & 1, & 1; \end{array}$$

eritque $a = 1$, $\Delta a = 1$, $p = n + 1$, $\Delta p = 1$,
unde ſumma quaeſita eſt;

$$S = \frac{x}{1-x}(1-(n+1)x^n) + \frac{x^2}{(1-x)^2}(1-x^n), \text{ ſeu}$$

$$S = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$$

II. *Quaeratur ſumma huius ſeriei finitae*

$$S = 1x + 4x^2 + 9x^3 + 16x^4 + \ldots\ldots + n^2x^n.$$

Inveſtigentur primum differentiae hoc modo:

$$\begin{array}{ccccc|ccc} 1, & 4, & 9, & 16, & \&c. & (n+1)^2, & (n+2)^2, & (n+3)^2, \&c. \\ & 3, & 5, & 7, & & & 2n+3, & 2n+5 \\ & & 2, & 2, & & & 2 & \end{array}$$

quibus inventis erit ſumma quaeſita $S =$

$$\frac{x}{1-x}(1-(n+1)^2x^n) + \frac{x^2}{(1-x)^2}(3-(2n+3)x^n) + \frac{x^3}{(1-x)^3}(2-2x^n),$$

ſeu $$S = \frac{x+xx-(n+1)^2x^{n+1}+(2nn+2n-1)x^{n+2}-nnx^{n+3}}{(1-x)^3},$$

6. Quodſi autem ſeries propoſita non eiuſmodi habeat coefficientes, qui tandem ad differentias conſtantes deducantur, tum tranſmutatio hic exhibita nihil confert ad eius ſummam determinandam. Neque vero etiam eius ope ſumma proxime definiri poterit commodius, quam per ipſam ſeriei propoſitae additionem fieri licet. Si enim in ſerie

$$ax + bx^2 + cx^3 + dx^4 + \&c.$$

fuerit $x < 1$ quo ſolo caſu ſummatio proprie ſic dicta locum habere poteſt, erit $\frac{x}{1-x} > x$, ideoque non ſeries minus convergit quam propoſita. Sin autem in ſerie propoſita fuerit

$x = 1$

$x = 1$ tum novae ſeriei omnes plane termini fiunt infiniti, quo ergo caſu iſta tranſmutatio nullius prorſus erit uſus.

7. Conſideremus autem ſeriem, in qua ſigna + & — alternatim ſe excipiant, quae ex praecedente deducetur ponendo x negativum. Si itaque fuerit

$$S = ax - bx^2 + cx^3 - dx^4 + ex^5 - \&c.$$

cuius ſeriei negativa oritur, ſi in praecedente ſtatuatur x negativum. Sumantur ergo ut ante differentiae $\Delta a, \Delta^2 a, \Delta^3 a$, &c. ex ſerie coefficientium a, b, c, d, e, &c. ſignis ad ſolas ipſius x poteſtates relatis, atque ſeries propoſita transformabitur in hanc: S =

$$\frac{x}{1+x}a - \frac{x^2}{(1+x)^2}\Delta a + \frac{x^3}{(1+x)^3}\Delta^2 a - \frac{x^4}{(1+x)^4}\Delta^3 a + \&c.$$

unde perſpicitur aequationem propoſitam iiſdem caſibus ſummari poſſe quibus praecedens. Scilicet ſi ſeries a, b, c, d, &c. tandem ad differentias conſtantes deducatur.

8. Hoc autem caſu iſta transformatio commodam praebet approximationem ad valorem ſeriei propoſitae:

$$ax - bx^2 + cx^3 - dx^4 + ex^5 - fx^6 + \&c.$$

quantuſcunque enim x fuerit numerus, fractio $\frac{x}{1+x}$ ſecundum cuius poteſtates altera ſeries progreditur, fit unitate minor: atque ſi ſit $x = 1$, erit $\frac{x}{1+x} = \frac{1}{2}$. Sin autem ſit $x < 1$ puta $x = \frac{1}{n}$ fiet $\frac{x}{1+x} = \frac{1}{n+1}$, ideoque ſeries per transformationem inventa ſemper magis convergit quam propoſita. Conſideremus imprimis caſum, quo $x = 1$, qui ad ſeries ſummandas ingens affert ſubſidium, ſitque

$$S = a - b + c - d + e - f + \&c.$$

ac denotentur differentiae primae, ſecundae & ſequentes ipſius a, quas progreſſio a, b, c, d, e, &c. praebet per

Δ,

Δa, $\Delta^2 a$, $\Delta^3 a$, &c. quibus inventis erit

$$S = \tfrac{1}{2} a - \tfrac{1}{4} \Delta a + \tfrac{1}{8} \Delta^2 a - \tfrac{1}{16} \Delta^3 a + \&c.$$

quae niſi actu terminatur, ſummam vero proximam ſatis commode exhibet.

9. Uſum igitur huius ultimae tranſmutationis, qua ſumſimus $x = 1$, in aliquot exemplis oſtendamus, ac primo quidem in eiuſmodi, quibus vera ſumma finite exprimi poteſt. Tales ſunt ſeries divergentes, quibus numeri a, b, c, d, &c. tandem ad differentias conſtantes deducant, quarum ſummae, cum recepto huius vocis ſignificatu, proprie non exhiberi queant, vocem ſummae hic eo ſenſu, quem ſupra tribuimus, accipimus, ita ut denotet valorem expreſſionis finitae, ex cuius evolutione propoſita ſeries naſcatur.

I. *Sit igitur propoſita haec ſeries Leibnitzii:*

$$S = 1 - 1 + 1 - 1 + 1 - 1 + \&c.$$

in qua cum omnes termini ſint aequales, fient omnes differentiae $= 0$, ideoque ob $a = 1$, erit $S = \tfrac{1}{2}$.

II. *Sit propoſita iſta ſeries:*

$$S = 1 - 2 + 3 - 4 + 5 - 6 + \&c.$$

Diff. I. $=$ 1, 1, 1, 1, 1, &c.

Cum ergo ſit $a = 1$, $\Delta a = 1$, erit $S = \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4}$.

III. *Sit propoſita haec ſeries:*

$$S = 1 - 3 + 5 - 7 + 9 - \&c.$$

Diff. I. $=$ 2, 2, 2, 2, &c.

Ob $a = 1$ & $\Delta a = 2$ ſit $S = \tfrac{1}{2} - \tfrac{2}{4} = 0$.

IV. *Sit propoſita haec ſeries trigonalium numerorum.*

$$S = 1 - 3 + 6 - 10 + 15 - 21 + \&c.$$

Diff. I. $=$ 2, 3, 4, 5, 6, &c.

Diff. II. $=$ 1, 1, 1, 1, &c.

Hic ergo ob $a = 1$, $\Delta a = 2$, & $\Delta\Delta a = 1$; erit

$$S = \tfrac{1}{2} - \tfrac{2}{4} + \tfrac{1}{8} = \tfrac{1}{8}.$$

V. *Sit propoſita ſeries quadratorum:*

$$S = 1 - 4 + 9 - 16 + 25 - 36 + \&c.$$

Diff. I. $=$ 3, 5, 7, 9, 11, &c.

Diff. II. $=$ 2, 2, 2, 2, &c.

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Ob $a = 1$; $\Delta a = 3$; $\Delta\Delta a = 2$; erit $S = \frac{1}{2} - \frac{3}{4} + \frac{2}{8} = 0$.

VI. *Sit propoſita ſeries biquadratorum:*

$$S = 1 - 16 + 81 - 256 + 625 - 1296 + \&c.$$

Diff. I. =	15,	65,	175,	369,	671	
Diff. II. =		50,	110,	194,	302	
Diff. III. =			60,	84,	108	
Diff. IV. =				24,	24	

Erit ergo $S = \frac{1}{2} - \frac{15}{4} + \frac{50}{8} - \frac{60}{16} + \frac{24}{32} = 0$.

10. Si ſeries magis divergant uti geometricae aliaeque ſimiles, eae hoc modo ſtatim in ſeriem magis convergentem tranſmutantur, quae niſi adhuc ſatis convergat, eodem modo in aliam magis convergentem convertetur.

I. *Sit propoſita haec ſeries geometrica:*

$$S = 1 - 2 + 4 - 8 + 16 - 32 + \&c.$$

Diff. I. =	1,	2,	4,	8,	16,	&c.
Diff. II. =		1,	2,	4,	8,	&c.
Diff. III. =			1,	2,	4,	&c.

Cum igitur in omnibus differentiis primus terminus ſit $= 1$. Summa ſeriei exprimetur hoc modo

$$S = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \&c.$$

cuius ſumma eſt $= \frac{1}{3}$, oritur enim ex evolutione fractionis $\frac{1}{2+1}$, dum propoſita oritur ex $\frac{1}{1+2}$.

II. *Sit propoſita haec ſeries recurrens:*

$$S = 1 - 2 + 5 - 12 + 29 - 70 + 169 - \&c.$$

Diff. I. =	1,	3,	7,	17,	41,	99	&c.
Diff. II. =		2,	4,	10,	24,	58	&c.
Diff. III. =			2,	6,	14,	34	&c.
Diff. IV. =				4,	8,	20	&c.
Diff. V. =					4,	12	&c.
Diff. VI. =						8	&c.
						&c.	

Continuarum ergo differentiarum termini primi conſtituunt hanc progreſſionem geometricam geminatam:

1,

1, 1, 2, 2, 4, 4, 8, 8, 16, 16, &c. unde erit

$$S = \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{2}{8} - \tfrac{2}{16} + \tfrac{4}{32} - \tfrac{4}{64} + \tfrac{8}{128} \text{ \&c.}$$

cum igitur praeter primum terminum reliqui bini ſe continuo deſtruant, erit $S = \frac{1}{2}$. Oritur autem ſeries propoſita ex evolutione fractionis $\frac{1}{1+2-1} = \frac{1}{2}$, uti in expreſſione naturae ſerierum recurrentium oſtendimus.

III. *Sit propoſita ſeries hypergeometrica:*

$$S = 1 - 2 + 6 - 24 + 120 - 720 + 5040 - \text{\&c.}$$

cuius differentias continuas hoc modo commodius inveſtigabimus:

	Diff. I.	Diff. II.	Diff. III.
1			
2	1		
6	4	3	
24	18	14	11
120	96	78	64
720	600	504	426
5040	4320	3720	3216 &c.
40320	35280	30960	27240
362880	322560	287280	256320
3628800	3265920	2943360	2656080.

Quibus differentiis ulterius continuatis erit:

$$S = \frac{1}{2} - \frac{1}{4} + \frac{3}{8} - \frac{11}{16} + \frac{53}{32} - \frac{309}{64} + \frac{2119}{128} - \frac{16687}{256}$$

$$+ \frac{148329}{512} - \frac{1468457}{1024} + \frac{16019531}{2048} - \frac{190899411}{4096} + \text{\&c.}$$

Colligantur duo termini initiales, eritque $S = \frac{1}{4} + A$

exiſtente $A = \frac{3}{8} - \frac{11}{16} + \frac{53}{32} - \frac{309}{64} + \frac{2119}{128} - \text{\&c.}$

Si nunc eodem modo differentiae capiantur, erit

$$A = \frac{3}{2^4} - \frac{5}{2^6} + \frac{21}{2^8} - \frac{99}{2^{10}} + \frac{615}{2^{12}} - \frac{4401}{2^{14}} + \frac{36585}{2^{16}}$$

$$- \frac{342207}{2^{18}} + \frac{3565323}{2^{20}} - \frac{40866525}{2^{22}} + \&c.$$

Colligantur duo termini initiales, quia convergunt, fietque

$$A = \frac{7}{2^6} + B \quad \text{exiſtente} \quad B = \frac{21}{2^8} - \frac{99}{2^{10}} + \&c.$$

cuius ſeriei differentiis denuo ſumendis fiet:

$$B = \frac{21}{2^9} - \frac{15}{2^{12}} + \frac{159}{2^{15}} - \frac{429}{2^{18}} + \frac{5241}{2^{21}} - \frac{26283}{2^{24}} + \frac{338835}{2^{27}}$$

$$- \frac{2771097}{2^{30}} + \&c.$$

Colligantur quatuor termini initiales in unum & ſtatuatur

$$B = \frac{153}{2^{12}} + \frac{843}{2^{18}} + C \quad \text{exiſtente} \quad C = \frac{5241}{2^{21}} - \frac{26283}{2^{24}} + \&c.$$

fietque aliquot terminis actu colligendis proxime:

$C = \frac{15645}{2^{24}} - \frac{60417}{2^{30}}$. Ex his ergo tandem concludetur ſumma ſeriei propoſitae: $S = 0,40082038$, quae tamen vix ultra tres quatuorve figuras pro accurata haberi poteſt ob nimiam ſeriei divergentiam, eſt tamen certe iuſto minor. Aliunde enim inveni hanc ſummam eſſe $= 0,4036524077$, ubi ne ultima quidem nota a vero aberrat.

11. Imprimis autem haec tranſmutatio ingentem affert utilitatem ad ſeries iam quidem, ſed lente convergentes in alias, quae multo promtius convergant, tranſmutandas. Quoniam vero termini ſequentes minores ſunt quam praecedentes, differentiae primae fiunt negativae; unde in ſequentibus ſignorum ratio probe eſt habenda.

I. *Sit propoſita haec ſeries:*

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \&c.$$

Diff.

Diff. I. $= -\frac{1}{2}; -\frac{1}{2.3}; -\frac{1}{3.4}; -\frac{1}{4.5}; -\frac{1}{5.6}$

Diff. II. $= +\frac{1}{3}; \frac{2}{2.3.4}; \frac{2}{3.4.5}; \frac{2}{4.5.6}$

Diff. III. $= -\frac{1}{4}; -\frac{2.3}{2.3.4.5}; -\frac{2.3}{3.4.5.6}$

Diff. IV. $= +\frac{1}{5}$; &c. Hinc ergo erit

$$S = \frac{1}{2} + \frac{1}{2.4} + \frac{1}{3.8} + \frac{1}{4.16} + \frac{1}{5.32} + \&c.$$

utramque autem hanc ſeriem logarithmum hyperbolicum binarii exhibere, iam in Introductione oſtendimus.

II. *Sit propoſita iſta ſeries pro circulo:*

$$S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \;\&c.$$

Diff. I. $= -\frac{2}{1.3}; -\frac{2}{3.5}; -\frac{2}{5.7}; -\frac{2}{7.9}; -\frac{2}{9.11}$ &c.

Diff. II. $= +\frac{2.4}{1.3.5}; \frac{2.4}{3.5.7}; \frac{2.4}{5.7.9}; \frac{2.4}{7.9.11}$ &c.

Diff. III. $= -\frac{2.4.6}{1.3.5.7}; -\frac{2.4.6}{3.5.7.9}; -$ &c.

&c.

Hinc ergo concluditur fore ſummam ſeriei:

$$S = \frac{1}{2} + \frac{1}{3.2} + \frac{1.2}{3.5.2} + \frac{1.2.3}{3.5.7.2} + \&c.$$

ſeu $\quad 2S = 1 + \frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \frac{1.2.3.4}{3.5.7.9} + \&c.$

III. *Quaeratur valor huius ſeriei infinitae*

$$S = l2 - l3 + l4 - l5 + l6 - l7 + l8 - l9 \;\&c.$$

Quia differentiae ab initio nimis fiunt inaequales, colligantur actu termini uſque ad $l10$ ex tabulis, quorum valor reperietur $= -0{,}3911005$; eritque

$$S = -0{,}3911005 + l10 - l11 + l12 - l13 + l14 - l15 + \&c.$$
in infinitum.

Deſumantur hi logarithmi ex tabulis, eorumque differentiae quaerantur hoc modo

	Diff. 1.	Diff. 2.	Diff. 3.	Diff. 4.	Diff. 5.
	+	−	+	−	+
$l10 = 1{,}0000000$					
$l11 = 1{,}0413927$	413927				
$l12 = 1{,}0791812$	377885	36042			
$l13 = 1{,}1139434$	347622	30263	5779		
$l14 = 1{,}1461280$	321846	25776	4487	1292	
$l15 = 1{,}1760913$	299633	22213	3563	924	368

Ex quibus reperitur

$$l10 - l11 + l12 - l13 + \&c. = \frac{1{,}0000000}{2} - \frac{413927}{4} - \frac{36042}{8} - \frac{5779}{16} - \frac{1292}{32} - \frac{368}{64}$$

$$= 0{,}4891606.$$

Hinc valor ſeriei propoſitae erit

$$S = l2 - l3 + l4 - l5 + \&c. = 0{,}0980601;$$

cui logarithmo reſpondet numerus $1{,}253315$.

12. Quemadmodum has tranſmutationes obtinuimus ponendo in ſerie loco x hanc fractionem $\frac{y}{1 \pm y}$, ita innumerabiles aliae tranſmutationes orientur, ſi loco x aliae functiones ipſius y ſubſtituantur. Sit iterum propoſita iſta ſeries:

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \&c.$$

atque ponatur $x = y(1 - y)$, quo facto ſeries orietur ſequens

$$\begin{aligned} S = ay &- ayy \\ &+ byy - 2by^3 + by^4 \\ &\qquad + cy^3 - 3cy^4 + 3cy^5 - cy^6 \\ &\qquad\qquad + dy^4 - 4dy^5 + 6dy^6 \\ &\qquad\qquad\qquad + ey^5 - 5ey^6 \quad \&c. \\ &\qquad\qquad\qquad\qquad + fy^6 \end{aligned}$$

Quodſi ergo altera harum ſerierum fuerit ſummabilis, ſimul alterius ſumma erit cognita. Ita ſi ſtatuatur

$$S = x + x^2 + x^3 + x^4 + x^5 + \&c. = \frac{x}{1-x}, \text{ erit}$$

$$S = y - y^3 - y^4 + y^6 + y^7 - y^9 - y^{10} + \&c.$$

Cuius ergo ſeriei ſumma erit $= \frac{y - yy}{1 - y + yy}$.

13. Si altera ſeries alicubi abrumpatur, tum ſumma prioris abſolute exhiberi poterit. Ponamus eſſe $a = 1$, & in ſerie inventa omnes terminos poſt primum evaneſcere, ut ſit $S = y$; ideoque ob $x = y - yy$, erit ſumma prioris $= \frac{1}{2} - \surd(\frac{1}{4} - x)$. Fiet autem ob $a = 1$; ut ſequitur:

$$b = \quad 1 = \frac{1}{4} \cdot 2^2$$

$$c = \quad 2 = \frac{1 \cdot 3}{4 \cdot 6} \cdot 2^4$$

$$d = \quad 5 = \frac{1 \cdot 3 \cdot 5}{4 \cdot 6 \cdot 8} \cdot 2^6$$

$$e = \quad 14 = \frac{1 \cdot 3 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8 \cdot 10} \cdot 2^8$$

$$f = \quad 42 = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} \cdot 2^{10}$$

$$g = 132 = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14} \cdot 2^{12}$$

&c.

unde prior ſeries abibit in hanc: $S = \frac{1}{2} - \surd(\frac{1}{4} - x) = x + x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 + 132x^7 + \&c.$ quae eadem invenitur, ſi quantitas ſurda $\surd(\frac{1}{4} - x)$ in ſeriem evolvatur, atque ab $\frac{1}{2}$ ſubtrahatur.

14. Statuamus, quo tranſmutatio latius pateat $x = y(1 + ny)^\nu$, atque ſeries propoſita:

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \&c.$$

tranſmutabitur in ſequentem: $S =$

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$$
\begin{array}{llllll}
S = ay + \frac{\nu}{1}nay^2 & + \frac{\nu(\nu-1)}{1.2}n^2ay^3 & + \frac{\nu(\nu-1)(\nu-2)}{1.\ 2.\ 3}n^3ay^4 & + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{1.\ 2.\ 3.\ 4}n^4ay^5 \\
\quad + by^2 & + \frac{2\nu}{1}nby^3 & + \frac{2\nu(2\nu-1)}{1.\ 2}n^2by^4 & + \frac{2\nu(2\nu-1)(2\nu-2)}{1.\ 2.\ 3}n^3by^5 \\
 & + cy^3 & + \frac{3\nu}{1}ncy^4 & + \frac{3\nu(3\nu-1)}{1.\ 2}n^2cy^5 \\
 & & + dy^4 & + \frac{4\nu}{1}ndy^5 \\
 & & & + ey^5 \\
 & & \&c. &
\end{array}
$$

Si ergo illius ſeriei ſumma fuerit cognita, & huius ſimul ſumma habebitur, ac viciſſim. Quoniam vero n & ν pro lubitu accipi poſſunt, hinc ex una ſerie ſummabili innumerae aliae ſummabiles inveniri poſſunt.

15. Poſſunt etiam eiuſmodi tranſmutationes fieri, ut ſeriei inventae ſumma fiat irrationalis, hoc modo.

Sit propoſita iſta ſeries:

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + fx^{11} + \&c. \quad \text{erit}$$

$$Sx = ax^2 + bx^4 + cx^6 + dx^8 + ex^{10} + fx^{12} + \&c.$$

Iam ſtatuatur $x = \frac{y}{\surd(1 - nyy)}$; erit $xx = \frac{yy}{1 - nyy}$,

atque ſeries propoſita tranſmutabitur in hanc: $\frac{Sy}{\surd(1 - nyy)} =$

$$
\begin{array}{lllll}
ay^2 + nay^4 & + n^2ay^6 & + n^3ay^8 & + n^4ay^{10} & + \&c. \\
\quad + by^4 & + 2nby^6 & + 3n^2by^8 & + 4n^3by^{10} & + \&c. \\
 & + cy^6 & + 3ncy^8 & + 6n^2cy^{10} & + \&c. \\
 & & + dy^8 & + 4ndy^{10} & + \&c. \\
 & & & + ey^{10} & + \&c.
\end{array}
$$

Si igitur ſumma S ex priori ſerie fuerit cognita, habebitur ſimul ſumma ſequentis ſeriei: $\frac{S}{\surd(1 - nyy)} =$

$$ay + (na+b)y^3 + (n^2a + 2nb + c)y^5 + (n^3a + 3n^2b + 3nc + d)y^7 + \&c.$$

16.

16. Si ſumatur $n = -1$; erunt coefficientes huius ſeriei differentiae continuae ipſius a, ex ſerie a, b, c, d, &c. ſin autem ſigna in ſerie propoſita alternentur, tum poſito $n = 1$ coefficientes erunt iſtae differentiae. Denotent ergo Δa, $\Delta^2 a$, $\Delta^3 a$, $\Delta^4 a$, &c. differentias primas, ſecundas, tertias, &c. ipſius a ex ſerie numerorum a, b, c, d, e, &c.

Ac ſi fuerit

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + \&c.$$

poſito $x = \frac{y}{\sqrt{(1+yy)}}$; erit

$$\frac{S}{\sqrt{(1+yy)}} = ay + \Delta a . y^3 + \Delta^2 a . y^5 + \Delta^3 a . y^7 + \&c.$$

Sin autem fuerit:

$$S = ax - bx^3 + cx^5 - dx^7 + ex^9 - \&c.$$

ponaturque $x = \frac{y}{\sqrt{(1-yy)}}$; erit.

$$\frac{S}{\sqrt{(1-yy)}} = ay - \Delta a . y^3 + \Delta^2 a . y^5 - \Delta^3 a . y^7 + \&c.$$

Quodſi ergo ſeries a, b, c, d, e, &c. tandem ad differentias conſtantes deducat, tum utraque ſeries abſolute ſummari poterit; quae ſummatio autem quoque ex ſuperioribus ſequitur.

17. Ponamus coefficientes a, b, c, d, &c. conſtituere hanc ſeriem $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$, &c. eritque, uti ſupra iam vidimus:

$$a = 1$$

$$\Delta a = -\frac{2}{3}.$$

$$\Delta^2 a = \frac{2.4}{3.5}$$

$$\Delta^3 a = -\frac{2.4.6}{3.5.7} \quad \&c.$$

unde ſequentes duas ſeries ſummabimus:

I. Sit $S = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \&c.$

Erit $S = \frac{1}{2} l \frac{1+x}{1-x}$; Poſito iam $x = \frac{y}{\sqrt{(1+yy)}}$,

fiet

$$S = \frac{1}{2} l \frac{\sqrt{(1+yy)}+y}{\sqrt{(1+yy)}-y} = l(\sqrt{(1+yy)}+y):$$

unde erit

$$\frac{l(\sqrt{(1+yy)}+y)}{\sqrt{(1+yy)}} = y - \frac{2}{3}y^3 + \frac{2.4}{3.5}y^5 - \frac{2.4.6}{3.5.7}y^7 + \&c.$$

II. Sit $S = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \&c.$

Erit $S = A \text{ tang } x$. Poſito iam $x = \frac{y}{\sqrt{(1-yy)}}$,

fiet

$$S = A \text{ tang } \frac{y}{\sqrt{(1-yy)}} = A \text{ ſin } y = A \text{ coſ} \sqrt{(1-yy)}.$$

Hancobrem obtinebitur iſta ſummatio:

$$\frac{A \text{ ſin } y}{\sqrt{(1-yy)}} = y + \frac{2}{3}y^3 + \frac{2.4}{3.5}y^5 + \frac{2.4.6}{3.5.7}y^7 + \&c.$$

18. Poſſunt quoque loco x functiones tranſcendentes ipſius y ſubſtitui, ſicque ſummationes aliae inventu difficiliores erui; verumtamen ne ſeries novae fiant nimis perplexae, eiuſmodi functiones eligi debent, quarum poteſtates facile exhiberi queant; quales ſunt quantitates exponentiales e^y. Propoſita igitur hac ſerie:

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \&c.$$

ponatur $x = e^{ny}y$, denotante e numerum, cuius logarithmus hyperbolicus $= 1$,

erit $x^2 = e^{2ny}y^2$; $x^3 = e^{3ny}y^3$; &c.

Generaliter vero eſt, uti conſtat:

$$e^z = 1 + z + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} + \&c.$$

Quare ſeries propoſita in hanc tranſmutabitur:

$$\begin{aligned} S = ay &+ 1nay^2 + \tfrac{1}{2}n^2ay^3 + \tfrac{1}{6}n^3ay^4 + \tfrac{1}{24}n^4ay^5 + \&c. \\ &+ by^2 + \tfrac{2}{1}nby^3 + \tfrac{4}{2}n^2by^4 + \tfrac{8}{6}n^3by^5 + \&c. \\ &\qquad + cy^3 + \tfrac{3}{1}ncy^4 + \tfrac{9}{2}n^2cy^5 + \&c. \\ &\qquad\qquad + dy^4 + \tfrac{4}{1}ndy^5 + \&c. \\ &\qquad\qquad\qquad + ey^5 + \&c. \end{aligned}$$

I. Sit ſeries propoſita geometrica :

$$S = x + x^2 + x^3 + x^4 + x^5 + \&c. \quad \text{erit} \quad S = \frac{x}{1 - x}.$$

Ponatur iam

$$n = -1; \quad \text{ut ſit} \quad x = e^{-y}y; \;\&\; S = \frac{e^{-y}y}{1 - e^{-y}y} = \frac{y}{e^y - y},$$

reperietur ſumma haec

$$\frac{y}{e^y - y} = y - \frac{1}{2}y^3 - \frac{1}{6}y^4 + \frac{5}{24}y^5 + \frac{19}{120}y^6 - \&c.$$

cuius autem ſeriei lex non perſpicitur.

II. Sint in altera ſerie omnes termini praeter primum $= 0$; erit $b = -na$; $c = \frac{3}{2}n^2a$; $d = -\frac{8}{3}n^3a$;

$$e = \frac{125}{24}n^4a; \quad f = -\frac{29}{30}n^5a; \quad \&c.$$

Cum ergo ſit ſumma $S = ay$; & $x = e^{ny}$; fiet:

$$y = x - nx^2 + \frac{3}{2}n^2x^3 - \frac{8}{3}n^3x^4 + \frac{125}{24}n^4x^5 - \frac{29}{30}n^5x^6 + \&c.$$

Quoniam vero in his ſeriebus lex progreſſionis non eſt manifeſta, ſummationes ex hac ſubſtitutione deductae parum habent utilitatis. Praecipue autem notari merentur transformationes ex ſubſtitutione $x = \frac{y}{1 \pm y}$ derivatae, quippe quae non ſolum eximias ſummationes, ſed etiam idoneos modos ad ſummas ſerierum appropinquandi ſuppeditant. His ergo, quae ſine calculi differentialis ope ſunt expedita, praemiſſis, ad ipſum huius calculi uſum in doctrina ſerierum oſtendendum progrediamur.

CAPUT II.

DE INVESTIGATIONE SERIERUM SUMMABILIUM.

19.

Si Seriei, in cuius terminis ineſt quantitas indeterminata x, ſumma fuerit cognita, quae utique erit functio ipſius x; tum quicunque valor ipſi x tribuatur, ſeriei ſumma perpetuo aſſignari poterit. Quare ſi loco x ponatur $x + dx$, ſeriei reſultantis ſumma erit aequalis ſummae prioris, una cum ipſius differentiali: unde ſequitur fore differentiale ſummae $=$ differentiali ſeriei. Quia vero hoc modo tam ſumma, quam ſinguli ſeriei termini multiplicati erunt per dx, ſi ubique per dx dividatur, habebitur nova ſeries, cuius ſumma erit cognita. Simili modo ſi haec ſeries cum ſua ſumma denuo differentietur, & ubique per dx dividatur, nova exoritur ſeries cum ſua ſumma: ſicque ex una ſerie ſummabili quantitatem indeterminatam x involvente, per continuam differentiationem innumerae novae ſeries pariter ſummabiles elicientur.

20. Quo haec clarius perſpiciantur, propoſita ſit progreſſio geometrica indeterminata, quippe cuius ſumma eſt cognita, haec:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \&c.$$

Si nunc differentiatio inſtituatur, erit:

$$\frac{dx}{(1-x)^2} = dx + 2xdx + 3x^2dx + 4x^3dx + 5x^4dx + \&c.$$

atque diviſione per dx inſtituta, habebitur:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \&c.$$

Si

Si denuo differentietur, atque per dx dividatur, prodibit:

$$\frac{2}{(1-x)^3} = 2 + 2.3x + 3.4x^2 + 4.5x^3 + 5.6x^4 + \&c. \quad \text{ſeu}$$

$$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + \&c.$$

ubi coefficientes ſunt numeri trigonales. Si haec porro differentietur, atque per $3dx$ dividatur, obtinebitur:

$$\frac{1}{(1-x)^4} = 1 + 4x + 10x^2 + 20x^3 + 35x^4 + \&c.$$

cuius coefficientes ſunt numeri pyramidales primi; Sicque ulterius procedendo oriuntur eaedem ſeries quas ex evolutione fractionis $\frac{1}{(1-x)^n}$ naſci conſtat.

21. Latius autem patebit haec ſerierum inveſtigatio, ſi antequam quaevis differentiatio ſuſcipiatur, ipſa ſeries una cum ſumma per quamvis ipſius x poteſtatem ſeu functionem multiplicetur. Sic cum ſit

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \&c.$$

multiplicetur ubique per x^m, eritque

$$\frac{x^m}{1-x} = x^m + x^{m+1} + x^{m+2} + x^{m+3} + x^{m+4} + \&c.$$

nunc differentietur haec ſeries, fietque per dx diviſo:

$$\frac{mx^{m-1} - (m-1)x^m}{(1-x)^2} = mx^{m-1} + (m+1)x^m$$
$$+ (m+2)x^{m+1} + (m+3)x^{m+2} + \&c.$$

Dividatur nunc per x^{m-1}, habebitur:

$$\frac{m-(m-1)x}{(1-x)^2} = \frac{m}{1-x} + \frac{x}{(1-x)^2} = m + (m+1)x$$
$$+ (m+2)x^2 + \&c.$$

multiplicetur haec antequam nova differentiatio ſuſcipiatur per x^n, ut ſit.

$$\frac{mx^n}{1-x}+\frac{x^{n+1}}{(1-x)^2}=mx^n+(m+1)x^{n+1}+(m+2)x^{n+2}+\text{\&c.}$$

Nunc inſtituatur differentiatio, & diviſo per dx erit:

$$\frac{mnx^{n-1}}{1-x}+\frac{(m+n+1)x^n}{(1-x)^2}+\frac{2x^{n+1}}{(1-x)^3}=mnx^{n-1}$$
$$+(m+1)(n+1)x^n+(m+2)(n+2)x^{n+1}+\text{\&c.}$$

Diviſione autem per x^{n-1} inſtituta, fiet

$$\frac{mn}{1-x}+\frac{(m+n+1)x}{(1-x)^2}+\frac{2xx}{(1-x)^3}=$$
$$mn+(m+1)(n+1)x+(m+2)(n+2)x^2+\text{\&c.}$$

ſicque ulterius progredi licebit: invenientur autem perpetuo eaedem ſeries, quae ex evolutione fractionum ſummam conſtituentium naſcuntur.

22. Quoniam progreſſionis geometricae primum aſſumtae ſumma ad quemvis terminum uſque aſſignari poteſt, hoc modo quoque ſeries definito terminorum numero conſtantes ſummabuntur. Ita cum ſit

$$\frac{x-x^{n+1}}{1-x}=x+x^2+x^3+x^4+\ldots\ldots+x^n$$

erit differentiatione inſtituta & terminis per dx diviſis

$$\frac{1}{(1-x)^2}-\frac{(n+1)x^n-nx^{n+1}}{(1-x)^2}=1+2x+3x^2+4x^3$$
$$+\ldots\ldots+nx^{n-1}$$

Hinc ſummae poteſtatum numerorum naturalium ad quemvis terminum inveniri poterunt. Multiplicetur enim haec ſeries per x, ut fiat:

$$\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}=x+2x^2+3x^3+\ldots+nx^n$$

quae denuo differentiata ac per dx diviſa dabit:

$$\frac{1+x-(n+1)^2x^n+(2nn+2n-1)x^{n+1}-nnx^{n+2}}{(1-x)^3}$$
$$=1+4x+9x^2+\ldots\ldots+n^2x^{n-1}$$

quae

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quae per x multiplicata dabit:

$$\frac{x+x^2-(n+1)^2x^{n+1}+(2nn+2n-1)x^{n+2}-nnx^{n+3}}{(1-x)^3}$$
$$=x+4x^2+9x^3+\ldots\ldots+n^2x^n$$

quae differentiata, per dx diviſa ac per x multiplicata producet ſeriem hanc:

$$x+8x^2+27x^3+\ldots\ldots+n^3x^n$$

cuius ſumma propterea invenietur. Ex hacque ſimili modo ſumma biquadratorum altiorumque poteſtatum indefinita eruetur.

23. Methodus igitur haec ad omnes ſeries quantitatem indeterminatam continentes accommodari poteſt, quarum quidem ſummae conſtant. Cum igitur praeter geometricas ſeries recurrentes omnes eadem praerogativa gaudeant, ut non ſolum in infinitum, ſed etiam ad quemvis terminum ſummari queant; ex iis quoque hac methodo innumerae aliae ſeries ſummabiles inveniri poterunt. Quod cum opus foret maxime diffuſum, ſi id perſequi vellemus, unicum caſum perpendamus. Sit ſcilicet propoſita haec ſeries:

$$\frac{x}{1-x-xx}=x+x^2+2x^3+3x^4+5x^5+8x^6+13x^7+\&c.$$

quae differentiata ac per dx diviſa dabit:

$$\frac{1+xx}{(1-x-xx)^2}=1+2x+6x^2+12x^3+25x^4+48x^5+91x^6+\&c.$$

Facile autem patet omnes has ſeries hoc modo reſultantes fore quoque recurrentes, quarum adeo ſummae ex ipſarum natura inveniri poterunt.

24. In genere igitur ſi ſeriei cuiuſpiam in hac forma contentae: $ax+bx^2+cx^3+dx^4+\&c.$ ſumma fuerit cognita, quam ponamus $=S$, eiuſdem ſeriei, ſi ſinguli termini ſingulatim per terminos progreſſionis arithmeticae multiplicentur, ſumma inveniri poterit. Sit enim

$$S=ax+bx^2+cx^3+dx^4+ex^5+\&c.$$

multiplicetur per x^m, erit

$$Sx^m=ax^{m+1}+bx^{m+2}+cx^{m+3}+dx^{m+4}+\&c.$$

differentietur haec aequatio, & dividatur per dx.

$$mSx^{m-1}+x^m\frac{dS}{dx}=(m+1)ax^m+(m+2)bx^{m+1}$$
$$+(m+3)cx^{m+2}+\&c.$$

dividatur per x^{m-1}, eritque

$$mS+\frac{xdS}{dx}=(m+1)ax+(m+2)bx^2+(m+3)cx^3+\&c.$$

Quodſi ergo huius ſequentis ſeriei ſumma deſideretur:
$\alpha ax+(\alpha+\beta)bx^2+(\alpha+2\beta)cx^3+(\alpha+3\beta)dx^4+$ &c.
multiplicetur ſuperior per β ac ſtatuatur $m\beta+\beta=\alpha$ ut ſit $m=\frac{\alpha-\beta}{\beta}$, eritque huius ſeriei ſumma $=(\alpha-\beta)S+\frac{\beta xdS}{dx}$.

25. Poterit etiam huius ſeriei propoſitae ſumma inveniri, ſi ſinguli eius termini multiplicentur per terminos ſeriei ſecundi ordinis ſingulatim, cuius ſcilicet differentiae demum ſecundae ſint conſtantes. Quoniam enim iam invenimus.

$$mS+\frac{xdS}{dx}=(m+1)ax+(m+2)bx^2+(m+3)cx^3+\&c.$$

multiplicetur per x^n, ut ſit

$$mSx^n+\frac{x^{n+1}dS}{dx}=(m+1)ax^{n+1}+(m+2)bx^{n+2}+\&c.$$

differentietur poſito dx conſtante, & per dx dividatur:

$$mnSx^{n-1}+\frac{(m+n+1)x^n dS}{dx}+\frac{x^{n+1}ddS}{dx^2}=$$
$$(m+1)(n+1)ax^n+(m+2)(n+2)bx^{n+1}+\&c.$$

Dividatur per x^{n-1}, ac multiplicetur per k, ut ſit

$$mnkS+\frac{(m+n+1)kxdS}{dx}+\frac{kx^2ddS}{dx^2}=$$
$$(m+1)(n+1)kax+(m+2)(n+2)kbx^2+(m+3)(n+3)kcx^3+\&c.$$

Comparetur nunc haec ſeries cum iſta:

erit

erit: | Diff. 1. | Diff. 2.

$$\left.\begin{array}{l} kmn+km+kn+k=\alpha \\ kmn+2km+2kn+4k=\alpha+\beta \\ kmn+3km+3kn+9k=\alpha+2\beta+\gamma \end{array}\right| \begin{array}{l} km+kn+3k=\beta \\ km+kn+5k=\beta+\gamma \end{array} \Big| \, 2k=\gamma$$

Ergo $k=\frac{1}{2}\gamma$; & $m+n=\frac{2\beta}{\gamma}-3$; atque

$$mn=\frac{\alpha}{k}-m-n-1=\frac{2\alpha}{\gamma}-\frac{2\beta}{\gamma}+2=\frac{2(\alpha-\beta+\gamma)}{\gamma}.$$

Hinc ſumma ſeriei quaeſita erit:

$$(\alpha-\beta+\gamma)S+\frac{(\beta-\gamma)xdS}{dx}+\frac{\gamma x^2 ddS}{2dx^2}.$$

26. Simili modo ſumma reperiri poterit ſeriei huius

$$Aa+Bbx+Ccx^2+Ddx^3+Eex^4+\&c.$$

ſi quidem cognita fuerit ſumma S ſeriei huius:

$$S=a+bx+cx^2+dx^3+ex^4+fx^5+\&c.$$

atque A, B, C, D, &c. conſtituant ſeriem, quae ad differentias conſtantes deducitur. Fingatur enim ſumma, quoniam eius forma ex antecedentibus colligitur, haec

$$\alpha S+\frac{\beta xdS}{dx}+\frac{\gamma x^2 ddS}{2dx^2}+\frac{\delta x^3 d^3S}{6dx^3}+\frac{\varepsilon x^4 d^4S}{24dx^4}+\&c.$$

Nunc ad litteras α, β, γ, δ, &c. inveniendas evolvantur ſingulae ſeries, eritque:

$$\begin{array}{lll} \alpha S & = \alpha a+\alpha bx+\alpha cx^2+\alpha dx^3+\alpha ex^4+\&c. \\ \frac{\beta xdS}{dx} & = \quad \beta bx+2\beta cx^2+3\beta dx^3+4\beta ex^4+\&c. \\ \frac{\gamma x^2 ddS}{2dx^2} & = \quad \gamma cx^2+3\gamma dx^3+6\gamma ex^4+\&c. \\ \frac{\delta x^3 d^3S}{6dx^3} & = \quad \delta dx^3+4\delta ex^4+\&c. \\ \frac{\varepsilon x^4 d^4S}{24dx^4} & = \quad \varepsilon ex^4+\&c. \end{array}$$

 quae

$$S = -0{,}3911005 + l10 - l11 + l12 - l13 + l14 - l15 + \&c.$$
in infinitum.

Deſumantur hi logarithmi ex tabulis, eorumque differentiae quaerantur hoc modo

	Diff. 1.	Diff. 2.	Diff. 3.	Diff. 4.	Diff. 5.
$l10 = 1{,}0000000$	+	−	+	−	+
$l11 = 1{,}0413927$	413927				
$l12 = 1{,}0791812$	377885	36042			
$l13 = 1{,}1139434$	347622	30263	5779		
$l14 = 1{,}1461280$	321846	25776	4487	1292	
$l15 = 1{,}1760913$	299633	22213	3563	924	368

Ex quibus reperitur

$$l10 - l11 + l12 - l13 + \&c. = \frac{1{,}0000000}{2} - \frac{413927}{4} - \frac{36042}{8} - \frac{5779}{16} - \frac{1292}{32} - \frac{368}{64}$$
$$= 0{,}4891606.$$

Hinc valor ſeriei propoſitae erit

$$S = l2 - l3 + l4 - l5 + \&c. = 0{,}0980601\,;$$

cui logarithmo reſpondet numerus $1{,}253315$.

12. Quemadmodum has tranſmutationes obtinuimus ponendo in ſerie loco x hanc fractionem $\frac{y}{1 \pm y}$, ita innumerabiles aliae tranſmutationes orientur, ſi loco x aliae functiones ipſius y ſubſtituantur. Sit iterum propoſita iſta ſeries:

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \&c.$$

atque ponatur $x = y(1-y)$, quo facto ſeries orietur ſequens

$$\begin{aligned} S = ay &- ayy \\ &+ byy - 2by^3 + by^4 \\ &\quad + cy^3 - 3cy^4 + 3cy^5 - cy^6 \\ &\quad\quad + dy^4 - 4dy^5 + 6dy^6 \\ &\quad\quad\quad + ey^5 - 5ey^6 \quad \&c. \\ &\quad\quad\quad\quad + fy^6 \end{aligned}$$

Quodſi ergo altera harum ſerierum fuerit ſummabilis, ſimul alterius ſumma erit cognita. Ita ſi ſtatuatur

$S = x + x^2 + x^3 + x^4 + x^5 + \&c. = \frac{x}{1-x}$, erit

$S = y - y^3 - y^4 + y^6 + y^7 - y^9 - y^{10} + \&c.$

Cuius ergo ſeriei ſumma erit $= \frac{y - yy}{1 - y + yy}$.

13. Si altera ſeries alicubi abrumpatur, tum ſumma prioris abſolute exhiberi poterit. Ponamus eſſe $a = 1$, & in ſerie inventa omnes terminos poſt primum evaneſcere, ut ſit $S = y$; ideoque ob $x = y - yy$, erit ſumma prioris $= \frac{1}{2} - \sqrt{(\frac{1}{4} - x)}$. Fiet autem ob $a = 1$; ut ſequitur:

$$b = \quad 1 = \frac{1}{4} \cdot 2^2$$

$$c = \quad 2 = \frac{1 \cdot 3}{4 \cdot 6} \cdot 2^4$$

$$d = \quad 5 = \frac{1 \cdot 3 \cdot 5}{4 \cdot 6 \cdot 8} \cdot 2^6$$

$$e = \quad 14 = \frac{1 \cdot 3 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8 \cdot 10} \cdot 2^8$$

$$f = \quad 42 = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} \cdot 2^{10}$$

$$g = 132 = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14} \cdot 2^{12}$$

&c.

unde prior ſeries abibit in hanc: $S = \frac{1}{2} - \sqrt{(\frac{1}{4} - x)} = x + x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 + 132x^7 + \&c.$ quae eadem invenitur, ſi quantitas ſurda $\sqrt{(\frac{1}{4} - x)}$ in ſeriem evolvatur, atque ab $\frac{1}{2}$ ſubtrahatur.

14. Statuamus, quo tranſmutatio latius pateat $x = y(1 + ny)^{\nu}$, atque ſeries propoſita:

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \&c.$$

tranſmutabitur in ſequentem: S =

$$\begin{array}{llllll}
S = ay + \frac{\nu}{1}nay^2 + & \frac{\nu(\nu-1)}{1.2}n^2ay^3 + & \frac{\nu(\nu-1)(\nu-2)}{1.\,2.\,3}n^3ay^4 + & \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{1.\,2.\,3.\,4}n^4ay^5 \\
+ \; by^2 + \frac{2\nu}{1}\,nby^3 + & \frac{2\nu(2\nu-1)}{1.\,2}\,n^2by^4 + & \frac{2\nu(2\nu-1)(2\nu-2)}{1.\,2.\,3}\,n^3by^5 & \\
+ \; cy^3 + & \frac{3\nu}{1}\,ncy^4 + & \frac{3\nu(3\nu-1)}{1.\,2}\,n^2cy^5 & \\
+ & dy^4 + & \frac{4\nu}{1}\,ndy^5 & \\
& + & ey^5 & \\
& \&c. & &
\end{array}$$

Si ergo illius ſeriei ſumma fuerit cognita, & huius ſimul ſumma habebitur, ac viciſſim. Quoniam vero n & ν pro lubitu accipi poſſunt, hinc ex una ſerie ſummabili innumerae aliae ſummabiles inveniri poſſunt.

15. Poſſunt etiam eiuſmodi tranſmutationes fieri, ut ſeriei inventae ſumma fiat irrationalis, hoc modo.

Sit propoſita iſta ſeries:

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + fx^{11} + \&c. \quad \text{erit}$$

$$Sx = ax^2 + bx^4 + cx^6 + dx^8 + ex^{10} + fx^{12} + \&c.$$

Iam ſtatuatur $x = \frac{y}{\sqrt{(1-nyy)}}$; erit $xx = \frac{yy}{1-nyy}$, atque ſeries propoſita tranſmutabitur in hanc: $\frac{Sy}{\sqrt{(1-nyy)}} =$

$$\begin{array}{llllll}
ay^2 + nay^4 + & n^2ay^6 + & n^3ay^8 + & n^4ay^{10} + \&c. \\
+ \; by^4 + & 2nby^6 + & 3n^2by^8 + & 4n^3by^{10} + \&c. \\
& + \; cy^6 + & 3ncy^8 + & 6n^2cy^{10} + \&c. \\
& + & dy^8 + & 4ndy^{10} + \&c. \\
& & + & ey^{10} + \&c.
\end{array}$$

Si igitur ſumma S ex priori ſerie fuerit cognita, habebitur ſimul ſumma ſequentis ſeriei: $\frac{S}{\sqrt{(1-nyy)}} =$

$$ay + (na+b)y^3 + (n^2a + 2nb + c)y^5 + (n^3a + 3n^2b + 3nc + d)y^7 + \&c.$$

16.

16. Si ſumatur $n = -1$; erunt coefficientes huius ſeriei differentiae continuae ipſius a, ex ſerie a, b, c, d, &c. ſin autem ſigna in ſerie propoſita alternentur, tum poſito $n = 1$ coefficientes erunt iſtae differentiae. Denotent ergo Δa, $\Delta^2 a$, $\Delta^3 a$, $\Delta^4 a$, &c. differentias primas, ſecundas, tertias, &c. ipſius a ex ſerie numerorum a, b, c, d, e, &c. Ac ſi fuerit

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + \&c.$$

poſito $x = \frac{y}{\sqrt{(1+yy)}}$; erit

$$\frac{S}{\sqrt{(1+yy)}} = ay + \Delta a.y^3 + \Delta^2 a.y^5 + \Delta^3 a.y^7 + \&c.$$

Sin autem fuerit:

$$S = ax - bx^3 + cx^5 - dx^7 + ex^9 - \&c.$$

ponaturque $x = \frac{y}{\sqrt{(1-yy)}}$; erit.

$$\frac{S}{\sqrt{(1-yy)}} = ay - \Delta a.y^3 + \Delta^2 a.y^5 - \Delta^3 a.y^7 + \&c.$$

Quodſi ergo ſeries a, b, c, d, e, &c. tandem ad differentias conſtantes deducat, tum utraque ſeries abſolute ſummari poterit; quae ſummatio autem quoque ex ſuperioribus ſequitur.

17. Ponamus coefficientes a, b, c, d, &c. conſtituere hanc ſeriem 1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, &c. eritque, uti ſupra iam vidimus:

$$a = 1$$

$$\Delta a = -\frac{2}{3}$$

$$\Delta^2 a = \frac{2.4}{3.5}$$

$$\Delta^3 a = -\frac{2.4.6}{3.5.7} \quad \&c.$$

unde ſequentes duas ſeries ſummabimus:

I. Sit $S = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \&c.$

Erit $S = \frac{1}{2} l \frac{1+x}{1-x}$. Poſito iam $x = \frac{y}{\surd(1+yy)}$, fiet

$$S = \frac{1}{2} l \frac{\surd(1+yy)+y}{\surd(1+yy)-y} = l(\surd(1+yy)+y):$$

unde erit

$$\frac{l(\surd(1+yy)+y)}{\surd(1+yy)} = y - \frac{2}{3}y^3 + \frac{2.4}{3.5}y^5 - \frac{2.4.6}{3.5.7}y^7 + \&c.$$

II. Sit $S = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \&c.$

Erit $S = A$ tang x. Poſito iam $x = \frac{y}{\surd(1-yy)}$, fiet

$$S = A \text{ tang } \frac{y}{\surd(1-yy)} = A \text{ ſin } y = A \text{ coſ } \surd(1-yy).$$

Hancobrem obtinebitur iſta ſummatio:

$$\frac{A \text{ ſin } y}{\surd(1-yy)} = y + \frac{2}{3}y^3 + \frac{2.4}{3.5}y^5 + \frac{2.4.6}{3.5.7}y^7 + \&c.$$

18. Poſſunt quoque loco x functiones tranſcendentes ipſius y ſubſtitui, ſicque ſummationes aliae inventu difficiliores erui; verumtamen ne ſeries novae fiant nimis perplexae, eiuſmodi functiones eligi debent, quarum poteſtates facile exhiberi queant; quales ſunt quantitates exponentiales e^y. Propoſita igitur hac ſerie:

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \&c.$$

ponatur $x = e^{ny}y$, denotante e numerum, cuius logarithmus hyperbolicus $= 1$,

erit $x^2 = e^{2ny}y^2$; $x^3 = e^{3ny}y^3$; &c.

Generaliter vero eſt, uti conſtat:

$$e^z = 1 + z + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} + \&c.$$

Quare ſeries propoſita in hanc tranſmutabitur:

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$$\begin{array}{rl} S = ay & + 1nay^2 + \tfrac{1}{2}n^2ay^3 + \tfrac{1}{6}n^3ay^4 + \tfrac{1}{24}n^4ay^5 + \&c. \\ & + by^2 + \tfrac{2}{1}nby^3 + \tfrac{4}{2}n^2by^4 + \tfrac{8}{6}n^3by^5 + \&c. \\ & \qquad + cy^3 + \tfrac{3}{1}ncy^4 + \tfrac{9}{2}n^2cy^5 + \&c. \\ & \qquad\qquad + dy^4 + \tfrac{4}{1}ndy^5 + \&c. \\ & \qquad\qquad\qquad + ey^5 + \&c. \end{array}$$

I. Sit ſeries propoſita geometrica:

$S = x + x^2 + x^3 + x^4 + x^5 + \&c.$ erit $S = \frac{x}{1-x}$.

Ponatur iam

$n = -1$; ut ſit $x = e^{-y}y$; & $S = \frac{e^{-y}y}{1-e^{-y}y} = \frac{y}{e^y - y}$,

reperietur ſumma haec

$$\frac{y}{e^y - y} = y - \frac{1}{2}y^3 - \frac{1}{6}y^4 + \frac{5}{24}y^5 + \frac{19}{120}y^6 - \&c.$$

cuius autem ſeriei lex non perſpicitur.

II. Sint in altera ſerie omnes termini praeter primum $= 0$; erit $b = -na$; $c = \frac{3}{2}n^2a$; $d = -\frac{8}{3}n^3a$;

$$e = \frac{125}{24}n^4a; \quad f = -\frac{29}{30}n^5a; \quad \&c.$$

Cum ergo ſit ſumma $S = ay$; & $x = e^{ny}$; fiet:

$$y = x - nx^2 + \frac{3}{2}n^2x^3 - \frac{8}{3}n^3x^4 + \frac{125}{24}n^4x^5 - \frac{29}{30}n^5x^6 + \&c.$$

Quoniam vero in his ſeriebus lex progreſſionis non eſt manifeſta, ſummationes ex hac ſubſtitutione deductae parum habent utilitatis. Praecipue autem notari merentur transformationes ex ſubſtitutione $x = \frac{y}{1 \pm y}$ derivatae, quippe quae non ſolum eximias ſummationes, ſed etiam idoneos modos ad ſummas ſerierum appropinquandi ſuppeditant. His ergo, quae ſine calculi differentialis ope ſunt expedita, praemiſſis, ad ipſum huius calculi uſum in doctrina ſerierum oſtendendum progrediamur.

CAPUT II.

DE INVESTIGATIONE SERIERUM SUMMABILIUM.

19.

Si Seriei, in cuius terminis ineſt quantitas indeterminata x, ſumma fuerit cognita, quae utique erit functio ipſius x; tum quicunque valor ipſi x tribuatur, ſeriei ſumma perpetuo aſſignari poterit. Quare ſi loco x ponatur $x + dx$, ſeriei reſultantis ſumma erit aequalis ſummae prioris, una cum ipſius differentiali: unde ſequitur fore differentiale ſummae = differentiali ſeriei. Quia vero hoc modo tam ſumma, quam ſinguli ſeriei termini multiplicati erunt per dx, ſi ubique per dx dividatur, habebitur nova ſeries, cuius ſumma erit cognita. Simili modo ſi haec ſeries cum ſua ſumma denuo differentietur, & ubique per dx dividatur, nova exoritur ſeries cum ſua ſumma: ſicque ex una ſerie ſummabili quantitatem indeterminatam x involvente, per continuam differentiationem innumerae novae ſeries pariter ſummabiles elicientur.

20. Quo haec clarius perſpiciantur, propoſita ſit progreſſio geometrica indeterminata, quippe cuius ſumma eſt cognita, haec:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \&c.$$

Si nunc differentiatio inſtituatur, erit:

$$\frac{dx}{(1-x)^2} = dx + 2xdx + 3x^2dx + 4x^3dx + 5x^4dx + \&c.$$

atque diviſione per dx inſtituta, habebitur:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \&c.$$

Si

Si denuo differentietur, atque per dx dividatur, prodibit:

$$\frac{2}{(1-x)^3} = 2 + 2.3x + 3.4x^2 + 4.5x^3 + 5.6x^4 + \&c. \quad \text{ſeu}$$

$$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + \&c.$$

ubi coefficientes ſunt numeri trigonales. Si haec porro differentietur, atque per $3dx$ dividatur, obtinebitur:

$$\frac{1}{(1-x)^4} = 1 + 4x + 10x^2 + 20x^3 + 35x^4 + \&c.$$

cuius coefficientes ſunt numeri pyramidales primi. Sicque ulterius procedendo oriuntur eaedem ſeries quas ex evolutione fractionis $\frac{1}{(1-x)^n}$ naſci conſtat.

21. Latius autem patebit haec ſerierum inveſtigatio, ſi antequam quaevis differentiatio ſuſcipiatur, ipſa ſeries una cum ſumma per quamvis ipſius x poteſtatem ſeu functionem multiplicetur. Sic cum ſit

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \&c.$$

multiplicetur ubique per x^m, eritque

$$\frac{x^m}{1-x} = x^m + x^{m+1} + x^{m+2} + x^{m+3} + x^{m+4} + \&c.$$

nunc differentietur haec ſeries, fietque per dx diviſo:

$$\frac{mx^{m-1} - (m-1)x^m}{(1-x)^2} = mx^{m-1} + (m+1)x^m$$
$$+ (m+2)x^{m+1} + (m+3)x^{m+2} + \&c.$$

Dividatur nunc per x^{m-1}, habebitur:

$$\frac{m-(m-1)x}{(1-x)^2} = \frac{m}{1-x} + \frac{x}{(1-x)^2} = m + (m+1)x$$
$$+ (m+2)x^2 + \&c.$$

multiplicetur haec antequam nova differentiatio ſuſcipiatur per x^n, ut ſit

$$\frac{mx^n}{1-x}+\frac{x^{n+1}}{(1-x)^2}=mx^n+(m+1)x^{n+1}+(m+2)x^{n+2}+\&c.$$

Nunc inſtituatur differentiatio, & diviſo per dx erit:

$$\frac{mnx^{n-1}}{1-x}+\frac{(m+n+1)x^n}{(1-x)^2}+\frac{2x^{n+1}}{(1-x)^3}=mnx^{n-1}$$
$$+(m+1)(n+1)x^n+(m+2)(n+2)x^{n+1}+\&c.$$

Diviſione autem per x^{n-1} inſtituta, fiet

$$\frac{mn}{1-x}+\frac{(m+n+1)x}{(1-x)^2}+\frac{2xx}{(1-x)^3}=$$
$$mn+(m+1)(n+1)x+(m+2)(n+2)x^2+\&c.$$

ſicque ulterius progredi licebit: invenientur autem perpetuo eaedem ſeries, quae ex evolutione fractionum ſummam conſtituentium naſcuntur.

22. Quoniam progreſſionis geometricae primum aſſumtae ſumma ad quemvis terminum uſque aſſignari poteſt, hoc modo quoque ſeries definito terminorum numero conſtantes ſummabuntur. Ita cum ſit

$$\frac{x-x^{n+1}}{1-x}=x+x^2+x^3+x^4+\ldots\ldots+x^n$$

erit differentiatione inſtituta & terminis per dx diviſis

$$\frac{1}{(1-x)^2}-\frac{(n+1)x^n-nx^{n+1}}{(1-x)^2}=1+2x+3x^2+4x^3$$
$$+\ldots\ldots+nx^{n-1}$$

Hinc ſummae poteſtatum numerorum naturalium ad quemvis terminum inveniri poterunt. Multiplicetur enim haec ſeries per x, ut fiat:

$$\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}=x+2x^2+3x^3+\ldots+nx^n$$

quae denuo differentiata ac per dx diviſa dabit:

$$\frac{1+x-(n+1)^2x^n+(2nn+2n-1)x^{n+1}-nnx^{n+2}}{(1-x)^3}$$
$$=1+4x+9x^2+\ldots\ldots+n^2x^{n-1}$$

quae

quae per x multiplicata dabit:

$$\frac{x + x^2 - (n+1)^2 x^{n+1} + (2nn + 2n - 1)x^{n+2} - nnx^{n+3}}{(1-x)^3}$$

$$= x + 4x^2 + 9x^3 + \dots\dots + n^2 x^n$$

quae differentiata, per dx diviſa ac per x multiplicata producet ſeriem hanc:

$$x + 8x^2 + 27x^3 + \dots\dots + n^3 x^n$$

cuius ſumma propterea invenietur. Ex hacque ſimili modo ſumma biquadratorum altiorumque poteſtatum indefinita eruetur.

23. Methodus igitur haec ad omnes ſeries quantitatem indeterminatam continentes accommodari poteſt, quarum quidem ſummae conſtant. Cum igitur praeter geometricas ſeries recurrentes omnes eadem praerogativa gaudeant, ut non ſolum in infinitum, ſed etiam ad quemvis terminum ſummari queant; ex iis quoque hac methodo innumerae aliae ſeries ſummabiles inveniri poterunt. Quod cum opus foret maxime diffuſum, ſi id perſequi vellemus, unicum caſum perpendamus.

Sit ſcilicet propoſita haec ſeries:

$$\frac{x}{1-x-xx} = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \&c.$$

quae differentiata ac per dx diviſa dabit:

$$\frac{1+xx}{(1-x-xx)^2} = 1 + 2x + 6x^2 + 12x^3 + 25x^4 + 48x^5 + 91x^6 + \&c.$$

Facile autem patet omnes has ſeries hoc modo reſultantes fore quoque recurrentes, quarum adeo ſummae ex ipſarum natura inveniri poterunt.

24. In genere igitur ſi ſeriei cuiuſpiam in hac forma contentae: $ax + bx^2 + cx^3 + dx^4 + \&c.$
ſumma fuerit cognita, quam ponamus $= S$, eiuſdem ſeriei, ſi ſinguli termini ſingulatim per terminos progreſſionis arithmeticae multiplicentur, ſumma inveniri poterit. Sit enim

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \&c.$$

multiplicetur per x^m, erit

$$Sx^m = ax^{m+1} + bx^{m+2} + cx^{m+3} + dx^{m+4} + \&c.$$

differentietur haec aequatio, & dividatur per dx.

$$mSx^{m-1}+x^m\frac{dS}{dx}=(m+1)ax^m+(m+2)bx^{m+1}$$
$$+(m+3)cx^{m+2}+\&c.$$

dividatur per x^{m-1}, eritque

$$mS+\frac{xdS}{dx}=(m+1)ax+(m+2)bx^2+(m+3)cx^3+\&c.$$

Quodſi ergo huius ſequentis ſeriei ſumma deſideretur:

$$\alpha ax+(\alpha+\beta)bx^2+(\alpha+2\beta)cx^3+(\alpha+3\beta)dx^4+\ \&c.$$

multiplicetur ſuperior per β ac ſtatuatur $m\beta+\beta=\alpha$ ut ſit $m=\frac{\alpha-\beta}{\beta}$, eritque huius ſeriei ſumma $=(\alpha-\beta)S+\frac{\beta xdS}{dx}$.

25. Poterit etiam huius ſeriei propoſitae ſumma inveniri, ſi ſinguli eius termini multiplicentur per terminos ſeriei ſecundi ordinis ſingulatim, cuius ſcilicet differentiae demum ſecundae ſint conſtantes. Quoniam enim iam invenimus.

$$mS+\frac{xdS}{dx}=(m+1)ax+(m+2)bx^2+(m+3)cx^3+\&c.$$

multiplicetur per x^n, ut ſit

$$mSx^n+\frac{x^{n+1}dS}{dx}=(m+1)ax^{n+1}+(m+2)bx^{n+2}+\&c.$$

differentietur poſito dx conſtante, & per dx dividatur:

$$mnSx^{n-1}+\frac{(m+n+1)x^n dS}{dx}+\frac{x^{n+1}ddS}{dx^2}=$$
$$(m+1)(n+1)ax^n+(m+2)(n+2)bx^{n+1}+\&c.$$

Dividatur per x^{n-1}, ac multiplicetur per k, ut ſit

$$mnkS+\frac{(m+n+1)kxdS}{dx}+\frac{kx^2ddS}{dx^2}=$$
$$(m+1)(n+1)kax+(m+2)(n+2)kbx^2+(m+3)(n+3)kcx^3+\&c.$$

Comparetur nunc haec ſeries cum iſta:

erit

erit:	Diff. 1.	Diff. 2.
$kmn + km + kn + k = \alpha$	$km + kn + 3k = \beta$	$2k = \gamma$
$kmn + 2km + 2kn + 4k = \alpha + \beta$	$km + kn + 5k = \beta + \gamma$	
$kmn + 3km + 3kn + 9k = \alpha + 2\beta + \gamma$		

Ergo $k = \frac{1}{2}\gamma$; & $m + n = \frac{2\beta}{\gamma} - 3$; atque

$$mn = \frac{\alpha}{k} - m - n - 1 = \frac{2\alpha}{\gamma} - \frac{2\beta}{\gamma} + 2 = \frac{2(\alpha - \beta + \gamma)}{\gamma}.$$

Hinc ſumma ſeriei quaeſita erit:

$$(\alpha - \beta + \gamma)S + \frac{(\beta - \gamma)xdS}{dx} + \frac{\gamma x^2 ddS}{2dx^2}.$$

26. Simili modo ſumma reperiri poterit ſeriei huius

$$Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \&c.$$

ſi quidem cognita fuerit ſumma S ſeriei huius:

$$S = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \&c.$$

atque A, B, C, D, &c. conſtituant ſeriem, quae ad differentias conſtantes deducitur. Fingatur enim ſumma, quoniam eius forma ex antecedentibus colligitur, haec

$$\alpha S + \frac{\beta xdS}{dx} + \frac{\gamma x^2 ddS}{2dx^2} + \frac{\delta x^3 d^3S}{6dx^3} + \frac{\varepsilon x^4 d^4S}{24dx^4} + \&c.$$

Nunc ad litteras α, β, γ, δ, &c. inveniendas evolvantur ſingulae ſeries, eritque:

$$\alpha S = \alpha a + \alpha bx + \alpha cx^2 + \alpha dx^3 + \alpha ex^4 + \&c.$$

$$\frac{\beta xdS}{dx} = \beta bx + 2\beta cx^2 + 3\beta dx^3 + 4\beta ex^4 + \&c.$$

$$\frac{\gamma x^2 ddS}{2dx^2} = \gamma cx^2 + 3\gamma dx^3 + 6\gamma ex^4 + \&c.$$

$$\frac{\delta x^3 d^3S}{6dx^3} = \delta dx^3 + 4\delta ex^4 + \&c.$$

$$\frac{\varepsilon x^4 d^4S}{24dx^4} = \varepsilon ex^4 + \&c.$$

 quae

quae ſimul ſumtae comparentur cum propoſita:

$$Z = Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \&c.$$

fietque comparatione ſingulorum terminorum inſtituta:

$$\alpha = A$$
$$\beta = B - \alpha = B - A$$
$$\gamma = C - 2\beta - \alpha = C - 2B + A$$
$$\delta = D - 3\gamma - 3\beta - \alpha = D - 3C + 3B - A$$
$$\&c.$$

His igitur valoribus inventis erit ſumma quaeſita:

$$Z = AS + \frac{(B - A)x\,dS}{1\,dx} + \frac{(C - 2B + A)x^2\,ddS}{1.2\,dx^2} + \frac{(D - 3C + 3B - A)x^3\,d^3S}{1.2.3\,dx^3} + \&c.$$

ſeu ſi ſeriei A, B, C, D, E, &c. differentiae continuae more conſueto indicentur, erit

$$Z = AS + \frac{\Delta A.x\,dS}{1\,dx} + \frac{\Delta^2 A.x^2\,d^2S}{1.2\,dx^2} + \frac{\Delta^3 A.x^3\,d^3S}{1.2.3\,dx^3} + \&c.$$

ſi quidem fuerit uti aſſumſimus:

$$S = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \&c.$$

Si ergo ſeries A, B, C, D, &c. tandem habeat differentias conſtantes, ſumma ſeriei Z finite exprimi poterit.

27. Quia ſumto e pro numero, cuius logarithmus hyperbolicus eſt $= 1$, eſt:

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} + \&c.$$

ſumatur haec ſeries pro priori, & cum ſit

$$S = e^x \quad \text{erit} \quad \frac{dS}{dx} = e^x; \quad \frac{ddS}{dx^2} = e^x; \quad \&c.$$

Quare huius ſeriei, quae ex illa & hac A, B, C, D, &c. componitur:

$$A + \frac{Bx}{1} + \frac{Cx^2}{1.2} + \frac{Dx^3}{1.2.3} + \frac{Ex^4}{1.2.3.4} + \&c.$$

ſum-

ſumma hoc modo exprimetur:

$$e^x\left(A+\frac{x\Delta A}{1}+\frac{xx\Delta^2 A}{1.2}+\frac{x^3\Delta^3 A}{1.2.3}+\frac{x^4\Delta^4 A}{1.2.3.4}+\&c.\right)$$

Sic ſi proponatur haec ſeries:

$$2+\frac{5x}{1}+\frac{10x^2}{1.2}+\frac{17x^3}{1.2.3}+\frac{26x^4}{1.2.3.4}+\frac{37x^5}{1.2.3.4.5}+\&c.$$

Ob ſeriem A, B, C, D, E, &c.

erit $A = 2,\ 5,\ 10,\ 17,\ 26$ &c.

$\Delta A = 3,\ 5,\ 7,\ 9$ &c.

$\Delta^2 A = 2,\ 2,\ 2$ &c.

erit huius ſeriei:

$$2+5x+\frac{10x^2}{2}+\frac{17x^3}{6}+\frac{26x^4}{24}+\&c.$$

ſumma $= e^x(2+3x+xx) = e^x(1+x)(2+x)$: quod quidem ſponte patet. Eſt enim

$$2e^x = 2+\frac{2x}{1}+\frac{2x^2}{2}+\frac{2x^3}{6}+\frac{2x^4}{24}+\&c.$$

$$3xe^x = 3x+\frac{3x^2}{1}+\frac{3x^3}{2}+\frac{3x^4}{6}+\&c.$$

$$xxe^x = xx+\frac{x^3}{1}+\frac{x^4}{2}+\&c.$$

$$e^x(2+3x+xx) = 2+5x+\frac{10xx}{2}+\frac{17x^3}{6}+\frac{26x^4}{24}+\&c.$$

28. Quae hactenus ſunt tradita non ſolum ad ſeries in infinitum excurrentes ſpectant, ſed etiam ad ſummas quotcunque terminorum: coefficientes enim a, b, c, d, &c. vel in infinitum progredi, vel ubicunque libuerit abrumpi poſſunt. Verum cum hoc non egeat uberiori explicatione, quae ex hactenus allatis ſequuntur, accuratius perpendamus. Propoſita ergo quacunque ſerie, cuius ſinguli termini duobus conſtent factoribus, quorum alteri ſeriem ad differentias conſtantes de-

ducentem conſtituant, huius ſeriei ſumma poterit aſſignari; dummodo omiſſis iſtis factoribus, ſeries fuerit ſummabilis.

Scilicet ſi propoſita ſit iſta ſeries

$$Z = Aa + Bbx + Ccxx + Ddx^3 + Eex^4 + \&c.$$

in qua quantitates A, B, C, D, E, &c. eiuſmodi ſeriem conſtituant, quae tandem ad differentias conſtantes perducatur; tum iſtius ſeriei ſumma exhiberi poterit, dummodo habeatur ſumma S huius ſeriei

$$S = a + bx + cx^2 + dx^3 + ex^4 + \&c.$$

Sumtis enim ex progreſſione A, B, C, D, E, &c. differentiis continuis, uti initio huius libri oſtendimus:

$$\begin{array}{ccccccc} A, & B, & C, & D, & E, & F, & \&c. \\ \Delta A, & \Delta B, & \Delta C, & \Delta D, & \Delta E, & \&c. & \\ \Delta^2 A, & \Delta^2 B, & \Delta^2 C, & \Delta^2 D, & \&c. & & \\ \Delta^3 A, & \Delta^3 B, & \Delta^3 C, & \&c. & & & \\ \Delta^4 A, & \Delta^4 B, & \&c. & & & & \\ \Delta^5 A, & \&c. & & & & & \\ \&c. & & & & & & \end{array}$$

erit ſeriei propoſitae ſumma

$$Z = SA + \frac{xdS}{1dx}\Delta A + \frac{x^2 ddS}{1.2dx^2}\Delta^2 A + \frac{x^3 d^3 S}{1.2.3dx^3}\Delta^3 A + \&c.$$

poſito in altioribus ipſius S differentialibus dx conſtante.

29. Si igitur ſeries A, B, C, D, &c. nunquam ad differentias conſtantes deducat, ſumma ſeriei Z per novam ſeriem infinitam exprimetur, quae interdum magis converget quam propoſita; ſique iſta ſeries in aliam ſibi aequalem transformabitur. Sit ad hoc declarandum propoſita haec ſeries:

$$Y = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \frac{y^5}{5} + \frac{y^6}{6} + \&c.$$

quam conſtat exprimere $l\frac{1}{1-y}$, ita ut ſit $Y = -l(1-y)$. Dividatur haec ſeries per y, & ſtatuatur $y = x$, & $Y = yZ$, ut ſit $Z = -\frac{1}{y}l(1-y) = -\frac{1}{x}l(1-x)$, erit

$$Z = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \&c.$$

quae comparata cum iſta:

$$S = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \&c. = \frac{1}{1-x},$$

dabit pro ſerie A, B, C, D, E, &c. hos valores:

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5}$$

$$-\frac{1}{1.2},\ -\frac{1}{2.3},\ -\frac{1}{3.4},\ -\frac{1}{4.5}$$

$$\frac{1.2}{1.2.3},\ \frac{1.2}{2.3.4},\ \frac{1.2}{3.4.5}$$

$$-\frac{1.2.3}{1.2.3.4},\ -\frac{1.2.3}{2.3.4.5}$$

&c.

Erit ergo $A = 1$; $\Delta A = -\frac{1}{2}$, $\Delta^2 A = \frac{1}{3}$, $\Delta^3 A = -\frac{1}{4}$ &c.

Porro cum ſit $S = \frac{1}{1-x}$, erit

$$\frac{dS}{dx} = \frac{1}{(1-x)^2}$$

$$\frac{ddS}{1.2\,dx^2} = \frac{1}{(1-x)^3}$$

$$\frac{d^3S}{1.2.3\,dx^3} = \frac{1}{(1-x)^4}$$

&c.

Quibus valoribus ſubſtitutis orietur ſumma: $Z =$

$$\frac{1}{1-x} - \frac{x}{2(1-x)^2} + \frac{x^2}{3(1-x)^3} - \frac{x^3}{4(1-x)^4} + \frac{x^4}{5(1-x)^5} - \&c.$$

Cum ergo ſit $x = y$, & $Y = -l(1-y) = yZ$; erit

$$-l(1-y) = \frac{y}{1-y} - \frac{y^2}{2(1-y)^2} + \frac{y^3}{3(1-y)^3} - \frac{y^4}{4(1-y)^4} + \&c.$$

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quae ſeries utique exprimit

$$l\left(1+\frac{y}{1-y}\right)=l\frac{1}{1-y}=-l(1-y),$$

cuius adeo veritas per ante demonſtrata conſtat.

30. Propoſita nunc ſit iſta ſeries, ut etiam uſus pateat, ſi poteſtates tantum impares occurrant, & ſigna alternentur.

$$Y=y-\frac{y^3}{3}+\frac{y^5}{5}-\frac{y^7}{7}+\frac{y^9}{9}-\frac{y^{11}}{11}+\&c.$$

ex qua conſtat eſſe $Y=A\ \mathrm{tang}\, y$.

Dividatur haec ſeries per y, & ponatur $\frac{Y}{y}=Z$,

& $yy=x$; erit:

$$Z=1-\frac{x}{3}+\frac{xx}{5}-\frac{x^3}{7}+\frac{x^4}{9}-\frac{x^5}{11}+\&c.$$

quae ſi comparetur cum iſta:

$S=1-x+xx-x^3+x^4-\&c.$ fiet $S=\frac{1}{1+x}$,

& ſeries coefficientium A, B, C, D, &c. fiet:

$$A=1,\ \frac{1}{3},\ \frac{1}{5},\ \frac{1}{7},\ \frac{1}{9}\ \&c.$$

$$\Delta A=-\frac{2}{3};\ -\frac{2}{3\cdot5};\ -\frac{2}{5\cdot7};\ -\frac{2}{7\cdot9}$$

$$\Delta^2 A=\frac{2\cdot4}{3\cdot5};\ \frac{2\cdot4}{3\cdot5\cdot7};\ \frac{2\cdot4}{5\cdot7\cdot9}$$

$$\Delta^3 A=-\frac{2\cdot4\cdot6}{3\cdot5\cdot7};\ -\frac{2\cdot4\cdot6}{3\cdot5\cdot7\cdot9}$$

$$\Delta^4 A=\frac{2\cdot4\cdot6\cdot8}{3\cdot5\cdot7\cdot9}$$

&c.

At

At cum ſit $S = \frac{1}{1+x}$; erit

$$\frac{dS}{1dx} = \frac{-1}{(1+x)^2};\ \frac{ddS}{1.2dx^2} = \frac{1}{(1+x)^3};\ \frac{d^3S}{1.2.3dx^3} = \frac{-1}{(1+x)^4}$$
&c.

Quare ſubſtitutis his valoribus fiet forma $Z =$

$$\frac{1}{1+x} + \frac{2x}{3(1+x)^2} + \frac{2.4x^2}{3.5(1+x)^3} + \frac{2.4.6x^3}{3.5.7(1+x)^4} + \&c.$$

Reſtituto ergo $x = yy$; & per y multiplicato fiet.

$$Y = A \text{ tang } y =$$

$$\frac{y}{1+yy} + \frac{2y^3}{3(1+yy)^2} + \frac{2.4.y^5}{3.5(1+yy)^3} + \frac{2.4.6.y^7}{3.5.7(1+yy)^4} + \&c.$$

31. Poteſt quoque ſuperior ſeries, qua arcus circuli per tangentem exprimitur, alio modo tranſmutari, eam comparando cum ſerie logarithmica.

Conſideremus nempe ſeriem

$$Z = 1 - \frac{x}{3} + \frac{xx}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \&c.$$

quam comparemus cum hac;

$$S = \frac{1}{0} - \frac{x}{2} + \frac{xx}{4} - \frac{x^3}{6} + \frac{x^4}{8} - \&c. = \frac{1}{0} - \frac{1}{2}\,l(1+x),$$

atque valores litterarum A, B, C, D, &c. erunt

$$A = \frac{0}{1};\ \frac{2}{3};\ \frac{4}{5};\ \frac{6}{7};\ \frac{8}{9};\ \&c.$$

$$\Delta A = \frac{2}{3};\ \frac{+2}{3.5};\ \frac{+2}{5.7};\ \frac{+2}{7.9};\ \&c.$$

$$\Delta^2 A = \frac{-2.4}{3.5};\ \frac{-2.4}{3.5.7};\ \frac{-2.4}{5.7.9};\ \&c.$$

$$\Delta^3 A = \frac{2.4.6}{3.5.7};\ \&c.$$

Dein-

Deinde cum ſit $S = \frac{1}{0} - \frac{1}{2} l(1+x)$; erit

$$\frac{dS}{1.dx} = -\frac{1}{2(1+x)}$$
$$\frac{ddS}{1.2dx^2} = +\frac{1}{4(1+x)^2}$$
$$\frac{d^3S}{1.2.3dx^3} = -\frac{1}{6(1+x)^3}$$
$$\frac{d^4S}{1.2.3.4dx^4} = \frac{1}{8(1+x)^4}$$
&c.

erit igitur $SA = S.\frac{0}{1} = 1$: & ex reliquis fiet:

$$Z = 1 - \frac{x}{3(1+x)} - \frac{2xx}{3.5(1+x)^2} - \frac{2.4x^3}{3.5.7(1+x)^3} - \&c.$$

Ponatur nunc $x = yy$, & multiplicetur per y fiet:

$$Y = A \text{ tang } y =$$
$$y - \frac{y^3}{3(1+yy)} - \frac{2y^5}{3.5(1+yy)^2} - \frac{2.4y^7}{3.5.7(1+yy)^3} - \&c.$$

Haec ergo tranſmutatio non impediebatur termino infinito $\frac{1}{0}$ qui in ſeriem S ingrediebatur. Sin autem cui ſuperſit dubium, is tantum ſingulos terminos praeter primum ſecundum poteſtates ipſius y in ſeries reſolvat, atque deprehendet actu ſeriem primum propoſitam reſultare.

32. Hactenus eiuſmodi tantum ſeries ſumus contemplati, in quibus omnes poteſtates variabilis occurrunt. Nunc igitur ad alias ſeries progrediamur, quae in ſingulis terminis eandem poteſtatem ipſius variabilis complectantur, cuiuſmodi eſt haec:

$$S = \frac{1}{a+x} + \frac{1}{b+x} + \frac{1}{c+x} + \frac{1}{d+x} + \&c.$$

Hu-

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Huius enim ſeriei ſi ſumma S fuerit cognita, ac per functionem quampiam ipſius x exprimatur, erit differentiando ac per $-dx$ dividendo:

$$\frac{-dS}{dx} = \frac{1}{(a+x)^2} + \frac{1}{(b+x)^2} + \frac{1}{(c+x)^2} + \frac{1}{(d+x)^2} + \&c.$$

Si haec ulterius differentietur, atque per $-2dx$ dividatur, cognoſcetur ſeries cuborum:

$$\frac{ddS}{2dx^2} = \frac{1}{(a+x)^3} + \frac{1}{(b+x)^3} + \frac{1}{(c+x)^3} + \frac{1}{(d+x)^3} + \&c.$$

haecque denuo differentiata, atque per $-3dx$ diviſa dabit:

$$\frac{-d^3S}{6dx^3} = \frac{1}{(a+x)^4} + \frac{1}{(b+x)^4} + \frac{1}{(c+x)^4} + \frac{1}{(d+x)^4} + \&c.$$

Similique modo omnium ſequentium poteſtatum ſummae reperientur, dummodo ſumma ſeriei primae fuerit cognita.

33. Huiuſmodi autem ſeries fractionum quantitatem indeterminatam involventes ſupra in introductione elicuimus, ubi oſtendimus, ſi circuli, cuius radius $=1$, ſemiperipheria ſtatuatur $=\pi$, fore

$$\frac{\pi}{n \sin \frac{m}{n}\pi} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \&c.$$

$$\frac{\pi \cos \frac{m}{n}\pi}{n \sin \frac{m}{n}\pi} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \&c.$$

Cum igitur pro m & n numeros quoſcunque aſſumere liceat, ſtatuamus $n=1$, & $m=x$; ut obtineamus ſeries illi quam in §. praec. propoſueramus ſimiles; hoc facto erit:

$$\frac{\pi}{\sin \pi x} = \frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} + \frac{1}{3-x} - \&c.$$

$$\frac{\pi \cos \pi x}{\sin \pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \&c.$$

Per differentiationes ergo ſummae quarumvis poteſtatum ex his fractionibus oriundarum exhiberi poterunt.

34. Conſideremus ſeriem priorem, ſitque brevitatis gratia $\frac{\pi}{\sin \pi x} = S$, cuius differentialia altiora capiantur poſito dx conſtante: eritque

$$S = \frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} + \frac{1}{3-x} - \&c.$$

$$\frac{-dS}{dx} = \frac{1}{xx} - \frac{1}{(1-x)^2} - \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} + \frac{1}{(2+x)^2} - \frac{1}{(3-x)^2} - \&c.$$

$$\frac{ddS}{2dx^2} = \frac{1}{x^3} + \frac{1}{(1-x)^3} - \frac{1}{(1+x)^3} - \frac{1}{(2-x)^3} + \frac{1}{(2+x)^3} + \frac{1}{(3-x)^3} - \&c.$$

$$\frac{-d^3S}{6dx^3} = \frac{1}{x^4} - \frac{1}{(1-x)^4} - \frac{1}{(1+x)^4} + \frac{1}{(2-x)^4} + \frac{1}{(2+x)^4} - \frac{1}{(3-x)^4} - \&c.$$

$$\frac{d^4S}{24dx^4} = \frac{1}{x^5} + \frac{1}{(1-x)^5} - \frac{1}{(1+x)^5} - \frac{1}{(2-x)^5} + \frac{1}{(2+x)^5} + \frac{1}{(3-x)^5} - \&c.$$

$$\frac{-d^5S}{120dx^5} = \frac{1}{x^6} - \frac{1}{(1-x)^6} - \frac{1}{(1+x)^6} + \frac{1}{(2-x)^6} + \frac{1}{(2+x)^6} - \frac{1}{(3-x)^6} - \&c.$$

&c.

ubi notandum eſt in poteſtatibus paribus ſigna eandem ſequi legem, pariterque in imparibus eandem legem ſignorum obſervari. Omnium ergo iſtarum ſerierum ſummae invenientur ex differentialibus expreſſionis $S = \frac{\pi}{\sin \pi x}$.

35. Ad Differentialia haec ſimplicius exprimenda ponamus

$\sin \pi x = p$ & $\cos \pi x = q$, erit

$dp = \pi dx \cos \pi x = \pi q dx$, & $dq = -\pi p dx$. Cum ergo ſit

$S = \frac{\pi}{p}$ erit

$-dS$

$$-\frac{dS}{dx}=\frac{\pi^2 q}{pp}$$

$$\frac{ddS}{dx^2}=\frac{\pi^3(pp+2qq)}{p^3}=\frac{\pi^3(qq+1)}{p^3} \quad \text{ob} \quad pp+qq=1$$

$$-\frac{d^3S}{dx^3}=\pi^4\left(\frac{5q}{pp}+\frac{6q^3}{p^4}\right)=\frac{\pi^4(q^3+5q)}{p^4}$$

$$\frac{d^4S}{dx^4}=\pi^5\left(\frac{24q^4}{p^5}+\frac{28q^2}{p^3}+\frac{5}{p}\right)=\frac{\pi^5(q^4+18q^2+5)}{p^5}$$

$$-\frac{d^5S}{dx^5}=\pi^6\left(\frac{120q^5}{p^6}+\frac{180q^3}{p^4}+\frac{61q}{pp}\right)=\frac{\pi^6(q^5+58q^3+61q)}{p^6}$$

$$\frac{d^6S}{dx^6}=\pi^7\left(\frac{720q^6}{p^7}+\frac{1320q^4}{p^5}+\frac{662q^2}{p^3}+\frac{61}{p}\right)$$

$$\text{vel}=\frac{\pi^7(q^6+179q^4+479q^2+61)}{p^7}$$

$$-\frac{d^7S}{dx^7}=\pi^8\left(\frac{5040q^7}{p^8}+\frac{10920q^5}{p^6}+\frac{7266q^3}{p^4}+\frac{1385q}{p^2}\right)$$

$$\text{vel}=\frac{\pi^8}{p^8}(q^7+543q^5+3111q^3+1385q)$$

$$\frac{d^8S}{dx^8}=\pi^9\left(\frac{40320q^8}{p^9}+\frac{100800q^6}{p^7}+\frac{83664q^4}{p^5}+\frac{24568q^2}{p^3}+\frac{1385}{p}\right)$$

Quae expreſſiones facile ulterius quouſque libuerit continuari poſſunt, ſi enim fuerit:

$$\pm\frac{d^nS}{dx^n}=\pi^{n+1}\left(\frac{\alpha q^n}{p^{n+1}}+\frac{\beta q^{n-2}}{p^{n-1}}+\frac{\gamma q^{n-4}}{p^{n-3}}+\frac{\delta q^{n-6}}{p^{n-5}}+\&c.\right)$$

erit differentiale ſequens ſignis mutatis:

$$\mp\frac{d^{n+1}S}{dx^{n+1}}=\pi^{n+2}\left(\frac{(n+1)\alpha q^{n+1}}{p^{n+2}}+\left.\begin{matrix}n\alpha\\+(n-1)\beta\end{matrix}\right\}\frac{q^{n-1}}{p^n}+\left.\begin{matrix}(n-2)\beta\\+(n-3)\gamma\end{matrix}\right\}\frac{q^{n-3}}{p^{n-2}}+\left.\begin{matrix}(n-4)\gamma\\+(n-5)\delta\end{matrix}\right\}\frac{q^{n-5}}{p^{n-4}}\ \&c.\right)$$

36. Ex his ergo obtinebuntur ſummae ſerierum ſuperiorum §. 34. exhibitarum ſequentes:

$$S = \pi \cdot \frac{1}{p}$$

$$\frac{-dS}{dx} = \frac{\pi^2}{1} \cdot \frac{q}{p^2}$$

$$\frac{ddS}{2dx^2} = \frac{\pi^3}{2}\left(\frac{2q^2}{p^3} + \frac{1}{p}\right)$$

$$\frac{-d^3S}{6dx^3} = \frac{\pi^4}{6}\left(\frac{6q^3}{p^4} + \frac{5q}{p^2}\right)$$

$$\frac{d^4S}{24dx^4} = \frac{\pi^5}{24}\left(\frac{24q^4}{p^5} + \frac{28q^2}{p^3} + \frac{5}{p}\right)$$

$$\frac{-d^5S}{120dx^5} = \frac{\pi^6}{120}\left(\frac{120q^5}{p^6} + \frac{180q^3}{p^4} + \frac{61q}{p^2}\right)$$

$$\frac{d^6S}{720dx^6} = \frac{\pi^7}{720}\left(\frac{720q^6}{p^7} + \frac{1320q^4}{p^5} + \frac{662q^2}{p^3} + \frac{61}{p}\right)$$

$$\frac{-d^7S}{5040dx^7} = \frac{\pi^8}{5040}\left(\frac{5040q^7}{p^8} + \frac{10920q^5}{p^6} + \frac{7266q^3}{p^4} + \frac{1385q}{p^2}\right)$$

$$\frac{d^8S}{40320dx^8} = \frac{\pi^9}{40320}\left(\frac{40320q^8}{p^9} + \frac{100800q^6}{p^7} + \frac{83664q^4}{p^5} + \frac{24568q^2}{p^3} + \frac{1385}{p}\right)$$

&c.

37. Tractemus ſimili modo alteram ſeriem ſupra inventam:

$$\frac{\pi \operatorname{coſ} \pi x}{\operatorname{ſin} \pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \&c.$$

atque poſito brevitatis ergo $\frac{\pi \operatorname{coſ} \pi x}{\operatorname{ſin} \pi x} = T$, orientur ſequentes ſummationes:

$$T = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \&c.$$

$$\frac{-dT}{dx} = \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} + \frac{1}{(2+x)^2} + \&c.$$

ddT

$$\frac{ddT}{2dx^2} = \frac{1}{x^3} - \frac{1}{(1-x)^3} + \frac{1}{(1+x)^3} - \frac{1}{(2-x)^3} + \frac{1}{(2+x)^3} - \&c.$$

$$\frac{-d^3T}{6dx^3} = \frac{1}{x^4} + \frac{1}{(1-x)^4} + \frac{1}{(1+x)^4} + \frac{1}{(2-x)^4} + \frac{1}{(2+x)^4} + \&c.$$

$$\frac{d^4T}{24dx^4} = \frac{1}{x^5} - \frac{1}{(1-x)^5} + \frac{1}{(1+x)^5} - \frac{1}{(2-x)^5} + \frac{1}{(2+x)^5} - \&c.$$

$$\frac{-d^5T}{120dx^5} = \frac{1}{x^6} + \frac{1}{(1-x)^6} + \frac{1}{(1+x)^6} + \frac{1}{(2-x)^6} + \frac{1}{(2+x)^6} + \&c.$$

&c.

ubi in poteſtatibus paribus omnes termini ſunt affirmativi, in imparibus autem ſigna + & — alternatim ſe excipiunt.

38. Quo differentialium horum valores innoteſcant, ponamus ut ante $\sin \pi x = p$ & $\operatorname{coſ} \pi x = q$, ut ſit $pp + qq = 1$; erit $dp = \pi q dx$ & $dq = -\pi p dx$. Quibus valoribus adhibitis erit:

$$T = \pi \cdot \frac{q}{p}$$

$$\frac{-dT}{dx} = \pi^2\left(\frac{qq}{pp} + 1\right) = \frac{\pi^2}{pp}$$

$$\frac{ddT}{dx^2} = \pi^3\left(\frac{2q^3}{p^3} + \frac{2q}{p}\right) = \frac{2\pi^3 q}{p^3}$$

$$\frac{-d^3T}{dx^3} = \pi^4\left(\frac{6q^4}{p^4} + \frac{8qq}{pp} + 2\right) = \pi^4\left(\frac{6qq}{p^4} + \frac{2}{pp}\right)$$

$$\frac{d^4T}{dx^4} = \pi^5\left(\frac{24q^3}{p^5} + \frac{16q}{p^3}\right)$$

$$\frac{-d^5T}{dx^5} = \pi^6\left(\frac{120q^4}{p^6} + \frac{120qq}{p^4} + \frac{16}{pp}\right)$$

$$\frac{d^6T}{dx^6} = \pi^7\left(\frac{720q^5}{p^7} + \frac{960q^3}{p^5} + \frac{272q}{p^3}\right)$$

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$$\frac{-d^7 T}{dx^7} = \pi^8 \left(\frac{5040 q^6}{p^8} + \frac{8400 q^4}{p^6} + \frac{3696 q^2}{p^4} + \frac{272}{p^2} \right)$$

$$\frac{d^8 T}{dx^8} = \pi^9 \left(\frac{40320 q^7}{p^9} + \frac{80640 q^5}{p^7} + \frac{48384 q^3}{p^5} + \frac{7936 q}{p^3} \right) \quad \text{\&c.}$$

Quae formulae facile ulterius quouſque libuerit continuari poſſunt. Si enim ſit

$$\pm \frac{d^n T}{dx^n} = \pi^{n+1} \left(\frac{\alpha q^{n-1}}{p^{n+1}} + \frac{\beta q^{n-3}}{p^{n-1}} + \frac{\gamma q^{n-5}}{p^{n-3}} + \frac{\delta q^{n-7}}{p^{n-5}} + \text{\&c.} \right)$$

erit expreſſio ſequens

$$\mp \frac{d^{n+1} T}{dx^{n+1}} = \pi^{n+2} \left(\frac{(n+1)\alpha q^n}{p^{n+2}} + \frac{(n-1)(\alpha+\beta) q^{n-2}}{p^n} + \frac{(n-3)(\beta+\gamma) q^{n-4}}{p^{n-2}} + \text{\&c.} \right)$$

39. Series ergo poteſtatum §. 37. datae ſequentes habebunt ſummmas poſito $\sin \pi x = p$ & $\cos \pi x = q$.

$$T = \pi \cdot \frac{q}{p}$$

$$\frac{-dT}{dx} = \pi^2 \frac{1}{pp}$$

$$\frac{ddT}{2dx^2} = \pi^3 \frac{q}{p^3}$$

$$\frac{-d^3 T}{6dx^3} = \pi^4 \left(\frac{qq}{p^4} + \frac{1}{3pp} \right)$$

$$\frac{d^4 T}{24dx^4} = \pi^5 \left(\frac{q^3}{p^5} + \frac{2q}{3p^3} \right)$$

$$\frac{-d^5 T}{120dx^5} = \pi^6 \left(\frac{q^4}{p^6} + \frac{3qq}{3p^4} + \frac{2}{15pp} \right)$$

$$\frac{d^6 T}{720dx^6} = \pi^7 \left(\frac{q^5}{p^7} + \frac{4q^3}{3p^5} + \frac{17q}{45p^3} \right)$$

$$\frac{-d^7 T}{5040dx^7} = \pi^8 \left(\frac{q^6}{p^8} + \frac{5q^4}{3p^6} + \frac{11q^2}{15p^4} + \frac{17}{315pp} \right)$$

$$\frac{d^8 T}{40320dx^8} = \pi^9 \left(\frac{q^7}{p^9} + \frac{6q^5}{3p^7} + \frac{6q^3}{5p^5} + \frac{62q}{315p^3} \right) \quad \text{\&c.}$$

40. Praeter has ſeries invenimus in introductione nonnullas alias, ex quibus ſimili modo per differentiationes novae elici poſſunt. Oſtendimus enim eſſe: $\frac{1}{2x} - \frac{\pi\surd x}{2x\ \mathrm{tang}\ \pi\surd x} =$

$$\frac{1}{1-x} + \frac{1}{4-x} + \frac{1}{9-x} + \frac{1}{16-x} + \frac{1}{25-x} + \&c.$$

Ponamus ſummam huius ſeriei eſſe $= S$,

ut ſit $S = \frac{1}{2x} - \frac{\pi}{2\surd x} \cdot \frac{\mathrm{cof}\ \pi\surd x}{\mathrm{fin}\ \pi\surd x}$, erit

$$\frac{dS}{dx} = -\frac{1}{2xx} + \frac{\pi}{4x\surd x} \cdot \frac{\mathrm{cof}\ \pi\surd x}{\mathrm{fin}\ \pi\surd x} + \frac{\pi\pi}{4x(\mathrm{fin}\ \pi\surd x)^2},$$

quae ergo expreſſio praebet ſummam huius ſeriei:

$$\frac{1}{(1-x)^2} + \frac{1}{(4-x)^2} + \frac{1}{(9-x)^2} + \frac{1}{(16-x)^2} + \frac{1}{(25-x)^2} + \&c.$$

Deinde quoque oſtendimus eſſe:

$$\frac{\pi}{2\surd x} \cdot \frac{e^{2\pi\surd x}+1}{e^{2\pi\surd x}-1} - \frac{1}{2x} =$$

$$\frac{1}{1+x} + \frac{1}{4+x} + \frac{1}{9+x} + \frac{1}{16+x} + \&c.$$

Quodſi ergo haec ſumma ponatur $= S$, erit:

$$\frac{-dS}{dx} = \frac{1}{(1+x)^2} + \frac{1}{(4+x)^2} + \frac{1}{(9+x)^2} + \frac{1}{(16+x)^2} + \&c.$$

At eſt

$$\frac{dS}{dx} = \frac{-\pi}{4x\surd x} \cdot \frac{e^{2\pi\surd x}+1}{e^{2\pi\surd x}-1} - \frac{\pi\pi}{x} \cdot \frac{e^{2\pi\surd x}}{(e^{2\pi\surd x}-1)^2} + \frac{1}{2xx}$$

Ergo ſumma huius ſeriei erit:

$$\frac{-dS}{dx} = \frac{\pi}{4x\surd x} \cdot \frac{e^{2\pi\surd x}+1}{e^{2\pi\surd x}-1} + \frac{\pi\pi}{x} \cdot \frac{e^{2\pi\surd x}}{(e^{2\pi\surd x}-1)^2} - \frac{1}{2xx}.$$

Similique modo ulterioribus differentiationibus ſummae ſequentium poteſtatum invenientur.

41.

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41. Si cognitus fuerit valor producti cuiuſpiam ex factoribus indeterminatam litteram involventibus compoſiti, ex eo per eandem methodum innumerabiles ſeries ſummabiles inveniri poterunt. Sit enim huius producti

$$(1+\alpha x)(1+\beta x)(1+\gamma x)(1+\delta x)(1+\varepsilon x) \text{ \&c.}$$

valor $= S$, functioni ſcilicet cuipiam ipſius x, erit logarithmis ſumendis:

$$lS = l(1+\alpha x) + l(1+\beta x) + l(1+\gamma x) + l(1+\delta x) + \text{\&c.}$$

Sumantur iam differentialia, erit diviſione per dx inſtituta:

$$\frac{dS}{Sdx} = \frac{\alpha}{1+\alpha x} + \frac{\beta}{1+\beta x} + \frac{\gamma}{1+\gamma x} + \frac{\delta}{1+\delta x} + \text{\&c.}$$

ex cuius ulteriori differentiatione ſummae poteſtatum quarumvis iſtarum fractionum reperietur; plane uti in exemplis praecedentibus fuſius expoſuimus.

42. Exhibuimus autem in introductione nonnullas iſtiuſmodi expreſſiones, ad quas hanc methodum accommodemus. Scilicet ſi ſit π arcus 180° circuli, cuius radius $=1$, oſtendimus eſſe:

$$\text{ſin}\,\frac{m\pi}{2n} = \frac{m\pi}{2n}\cdot\frac{4nn-mm}{4nn}\cdot\frac{16nn-mm}{16nn}\cdot\frac{36nn-mm}{36nn} \text{ \&c.}$$

$$\text{coſ}\,\frac{m\pi}{2n} = \frac{nn-mm}{nn}\cdot\frac{9nn-mm}{9nn}\cdot\frac{25nn-mm}{25nn}\cdot\frac{49nn-mm}{49nn} \text{ \&c.}$$

Ponamus $n=1$ & $m=2x$; ut ſit

$$\text{ſin}\,\pi x = \pi x\cdot\frac{1-xx}{1}\cdot\frac{4-xx}{4}\cdot\frac{9-xx}{9}\cdot\frac{16-xx}{16}\cdot\text{\&c. vel}$$

$$\text{ſin}\,\pi x = \pi x\cdot\frac{1-x}{1}\cdot\frac{1+x}{1}\cdot\frac{2-x}{2}\cdot\frac{2+x}{2}\cdot\frac{3-x}{3}\cdot\frac{3+x}{3}\cdot\frac{4-x}{4}\cdot\text{\&c.} \quad \&$$

$$\text{coſ}\,\pi x = \frac{1-4xx}{1}\cdot\frac{9-4xx}{9}\cdot\frac{25-4xx}{25}\cdot\frac{49-4xx}{49}\cdot\text{\&c.} \quad \text{ſeu}$$

$$\text{coſ}\,\pi x = \frac{1-2x}{1}\cdot\frac{1+2x}{1}\cdot\frac{3-2x}{3}\cdot\frac{3+2x}{3}\cdot\frac{5-2x}{5}\cdot\frac{5+2x}{5}\cdot\text{\&c.}$$

Ex

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Ex his ergo expreſſionibus, ſi logarithmi ſumantur, erit:

$$l\,\text{ſin}\,\pi x = l\,\pi x + l\,\frac{1-x}{1} + l\,\frac{1+x}{1} + l\,\frac{2-x}{2} + l\,\frac{2+x}{2} + l\,\frac{3-x}{3} + \&c.$$

$$l\,\text{coſ}\,\pi x = l\,\frac{1-2x}{1} + l\,\frac{1+2x}{1} + l\,\frac{3-2x}{3} + l\,\frac{3+2x}{3} + l\,\frac{5-2x}{5} + \&c.$$

43. Sumamus nunc harum ſerierum logarithmicarum differentialia, & diviſione ubique per dx facta prior ſeries dabit:

$$\frac{\pi\,\text{coſ}\,\pi x}{\text{ſin}\,\pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \&c.$$

quae eſt ea ipſa ſeries, quam §. 37. tractavimus. Altera vero ſeries dabit:

$$\frac{-\pi\,\text{ſin}\,\pi x}{\text{coſ}\,\pi x} = \frac{-2}{1-2x} + \frac{2}{1+2x} - \frac{2}{3-2x} + \frac{2}{3+2x} - \frac{2}{5-2x} + \&c.$$

Ponamus $2x = z$, ut ſit $x = \frac{z}{2}$, & dividamus per -2 erit:

$$\frac{\pi\,\text{ſin}\,\frac{1}{2}\pi z}{2\,\text{coſ}\,\frac{1}{2}\pi z} = \frac{1}{1-z} - \frac{1}{1+z} + \frac{1}{3-z} - \frac{1}{3+z} + \frac{1}{5-z} - \&c.$$

Cum autem ſit

$$\text{ſin}\,\tfrac{1}{2}\pi z = \surd\,\frac{1-\text{coſ}\,\pi z}{2} \quad \& \quad \text{coſ}\,\tfrac{1}{2}\pi z = \surd\,\frac{1+\text{coſ}\,\pi z}{2}$$

erit:

$$\frac{\pi\surd(1-\text{coſ}\,\pi z)}{\surd(1+\text{coſ}\,\pi z)} = \frac{2}{1-z} - \frac{2}{1+z} + \frac{2}{3-z} - \frac{2}{3+z} + \frac{2}{5-z} - \&c.$$

ſeu loco z ſcribendo x erit:

$$\frac{\pi\surd(1-\text{coſ}\,\pi x)}{\surd(1+\text{coſ}\,\pi x)} = \frac{2}{1-x} - \frac{2}{1+x} + \frac{2}{3-x} - \frac{2}{3+x} + \frac{2}{5-x} - \&c.$$

Addatur haec ſeries ad primum inventam:

$$\frac{\pi\,\text{coſ}\,\pi x}{\text{ſin}\,\pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \&c.$$

Mm Atque

Atque reperietur huius ſeriei:

$$\frac{1}{x}+\frac{1}{1-x}-\frac{1}{1+x}-\frac{1}{2-x}+\frac{1}{2+x}+\frac{1}{3-x}-\frac{1}{3+x}-\&c.$$

Summa $=\frac{\pi\surd(1-\text{coſ}\,\pi x)}{\surd(1+\text{coſ}\,\pi x)}+\frac{\pi\,\text{coſ}\,\pi x}{\text{ſin}\,\pi x}$. At fractio haec $\frac{\surd(1-\text{coſ}\,\pi x)}{\surd(1+\text{coſ}\,\pi x)}$, ſi numerator & denominator multiplicetur per $\surd(1-\text{coſ}\,\pi x)$ abit in $\frac{1-\text{coſ}\,\pi x}{\text{ſin}\,\pi x}$. Quocirca ſumma ſeriei erit $=\frac{\pi}{\text{ſin}\,\pi x}$, quae eſt ea ipſa, quam §. 34. habuimus: unde eam ulterius non perſequemur.

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CAPUT III.

DE INVENTIONE DIFFERENTIARUM FINITARUM

44.

Quemadmodum ex functionum differentiis finitis earum differentialia facile inveniri queant, in initio fuſius expoſuimus, atque adeo ex hoc fonte principium differentialium derivavimus. Si enim differentiae, quae aſſumtae erant finitae, evaneſcant, in nihilumque abeant, oriuntur differentialia; & quia hoc caſu plures & ſaepe innumeri termini, qui differentiam finitam conſtituunt, reiiciuntur, differentialia multo facilius inveniri, atque commodius ſuccinctiuſque exprimi poſſunt, quam differentiae finitae. Neque igitur hinc viciſſim via patere videtur, a differentialibus ad differentias finitas aſcendendi. Interim tamen eo modo, quo hic utemur, ex differentialibus omnium ordinum cuiuſcunque functionis, eiuſdem differentiae finitae omnes definiri poterunt.

45. Sit y functio quaecunque ipſius x, quae cum poſito $x + dx$ loco x abeat in $y + dy$, ſi denuo loco x ponatur $x + dx$, valor $y + dy$ ſuo differentiali $dy + ddy$ augebitur, fietque $= y + 2dy + ddy$, qui ergo valor reſpondebit ipſius x valori $x + 2dx$. Simili modo ſi ponamus quantitatem x continuo ſuo differentiali dx augeri, ut ſucceſſive valores

$$x + dx;\ x + 2dx;\ x + 3dx;\ x + 4dx;\ \&c.$$

induat, valores ipſius y reſpondentes erunt, quos haec tabella indicat:

Valores ipſius	Valores reſpondentes functionis
x	y
$x + dx$	$y + dy$
$x + 2dx$	$y + 2dy + ddy$
$x + 3dx$	$y + 3dy + 3ddy + d^3y$
$x + 4dx$	$y + 4dy + 6ddy + 4d^3y + d^4y$
$x + 5dx$	$y + 5dy + 10ddy + 10d^3y + 5d^4y + d^5y$
$x + 6dx$	$y + 6dy + 15ddy + 20d^3y + 15d^4y + 6d^5y + d^6y$
&c.	&c.

46. Generaliter ergo ſi x abeat in $x + ndx$, functio y recipiet hanc formam:

$$y + \frac{n}{1}\,dy + \frac{n(n-1)}{1.\ 2}\,ddy + \frac{n(n-1)(n-2)}{1.\ 2.\ 3}\,d^3y + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}\,d^4y + \&c.$$

in qua expreſſione, etſi quilibet terminus infinities minor eſt quam praecedens, tamen nullum praetermiſimus, quo iſta formula ad praeſens negotium apta redderetur. Statuemus enim pro n numerum infinite magnum, & quoniam notavimus, fieri poſſe ut productum ex quantitate infinite magna in infinite parvam aequetur quantitati finitae, terminus ſecundus utique homogeneus fieri poterit primo, ſeu ndy quantitatem finitam repraeſentare poterit. Ob eandemque rationem terminus tertius $\frac{n(n-1)}{1.\ 2}\,ddy$, etſi ddy infinities minus eſt quam dy, tamen quia alter factor $\frac{n(n-1)}{1.\ 2}$ infinities maior eſt quam n, terminus quoque tertius quantitatem finitam exprimere poterit: ſicque poſito n numero infinito nullum illius expreſſionis terminum reiicere licebit.

47. Poſito autem n numero infinito quocunque is nume-

mero finito ſive augeatur ſive diminuatur, numerus reſultans ad n habebit rationem aequalitatis, hincque pro ſingulis factoribus $n-1$, $n-2$, $n-3$, $n-4$, &c. ubique ſcribi poterit n. Cum enim ſit

$$\frac{n(n-1)}{1.2}ddy = \tfrac{1}{2}nnddy - \tfrac{1}{2}nddy$$

prior terminus $\frac{1}{2}nnddy$ ad poſteriorem $\frac{1}{2}nddy$ rationem tenebit ut n ad 1, ſicque hic reſpectu illius evaneſcet; loco $\frac{n(n-1)}{1.2}$ ergo ſcribi poterit $\frac{1}{2}nn$. Simili modo quarti termini coefficiens $\frac{n(n-1)(n-2)}{1.2.3}$ contrahi poterit in $\frac{n^3}{6}$ pariterque in ſequentibus numeri, quibus n in factoribus diminuitur, negligi poterunt. Hoc vero facto functio y, ſi loco x ponatur $x+ndx$, exiſtente numero n infinito, ſequentem valorem accipiet:

$$y + \frac{ndy}{1} + \frac{nnddy}{1.2} + \frac{n^3d^3y}{1.2.3} + \frac{n^4d^4y}{1.2.3.4} + \frac{n^5d^5y}{1.2.3.4.5} + \&c.$$

48. Cum igitur ſumto n numero infinite magno etiamſi dx ſit infinite parvum, productum ndx quantitatem finitam exprimere poſſit, ponamus $ndx = \omega$, ut ſit $n = \frac{\omega}{dx}$ erit utique n numerus infinitus, cum ſit quotus ex diviſione quantitatis finitae ω, per infinite parvam dx reſultans. Valore autem hoc loco n adhibito cognoſcemus, ſi quantitas variabilis x augeatur quavis quantitate finita ω, ſeu ſi loco x ponatur $x+\omega$, tum quamvis ipſius functionem y abituram eſſe in hanc formam:

$$y + \frac{\omega dy}{1dx} + \frac{\omega^2 ddy}{1.2dx^2} + \frac{\omega^3 d^3y}{1.2.3dx^3} + \frac{\omega^4 d^4y}{1.2.3.4dx^4} + \&c.$$

cuius expreſſionis ſinguli termini per continuam ipſius y differentiationem inveniri poterunt. Cum enim y ſit functio ipſius

x,

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x, oſtendimus ſupra, has functiones omnes $\frac{dy}{dx}$; $\frac{ddy}{dx^2}$; $\frac{d^3y}{dx^3}$; &c. quantitates finitas exhibere.

49. Cum igitur, dum quantitas variabilis x quantitate finita ω augeri aſſumitur, functio eius quaecunque y augeatur ſua differentia prima, quam ſupra per Δy indicavimus, exiſtente $\omega = \Delta x$: differentia ipſius y per continuam differentiationem reperiri poterit; erit enim:

$$\Delta y = \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \&c. \quad \text{ſeu}$$

$$\Delta y = \frac{\Delta x}{1} \cdot \frac{dy}{dx} + \frac{\Delta x^2}{2} \cdot \frac{ddy}{dx^2} + \frac{\Delta x^3}{6} \cdot \frac{d^3y}{dx^3} + \frac{\Delta x^4}{24} \cdot \frac{d^4y}{dx^4} + \&c.$$

Sicque differentia finita Δy exprimitur per progreſſionem; cuius ſinguli termini ſecundum poteſtates ipſius Δx procedunt. Atque hinc viciſſim patet, ſi quantitas x tantum quantitate infinite parva augeatur, ut Δx abeat in eius differentiale dx, omnes terminos prae primo evaneſcere, foreque $\Delta y = dy$; facto enim $\Delta x = dx$, differentia Δy abit per definitionem in differentiale dy.

50. Quoniam ſi loco x ponatur $x + \omega$, eius functio quaecunque y induit ſequentem valorem:

$$y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \&c.$$

veritas huius expreſſionis comprobari poterit eiuſmodi exemplis, quibus differentialia altiora ipſius y tandem evaneſcunt: his enim caſibus numerus terminorum ſuperioris expreſſionis fiet finitus:

EXEMPLUM I.

Quaeratur valor expreſſionis $xx - x$ *ſi loco* x *ponatur* $x + 1$.

Ponatur $y = xx - x$; & cum x in $x + 1$ abire ſtatuatur, fiet $\omega = 1$, ſumtis iam differentialibus erit:

dy

$\frac{dy}{dx} = 2x - 1$; $\frac{ddy}{dx^2} = 2$; $\frac{d^3y}{dx^3} = 0$; &c.

Hinc functio $y = xx - x$ poſito $x + 1$ loco x abibit in: $xx - x + 1(2x - 1) + \frac{1}{2}.2 = xx + x$.

Quodſi autem in $xx - x$ loco x actu ponatur $x + 1$ abibit

$$\begin{array}{lcl} xx & \text{in} & xx + 2x + 1 \\ x & \text{in} & x + 1 \\ \hline \end{array}$$

Ergo $xx - x$ in $xx + x$.

EXEMPLUM II.

Quaeratur valor expreſſionis $x^3 + xx + x$, *ſi loco* x *ponatur* $x + 2$.

Ponatur $y = x^3 + xx + x$, fietque $\omega = 2$; nunc cum ſit $y = x^3 + xx + x$

erit
$$\frac{dy}{dx} = 3xx + 2x + 1$$
$$\frac{ddy}{dx^2} = 6x + 2$$
$$\frac{d^3y}{dx^3} = 6$$
$$\frac{d^4y}{dx^4} = 0.$$

Ex his valor functionis $y = x^3 + xx + x$, ſi pro x ſtatuatur $x + 2$, erit ſequens:

$x^3 + xx + x + 2(3xx + 2x + 1) + \frac{4}{2}(6x + 2) + \frac{8}{6}.6 = x^3 + 7xx + 17x + 14$, qui idem prodit ſi actu loco x ſubſtituatur $x + 2$.

EXEMPLUM III.

Quaeratur valor expreſſionis $xx + 3x + 1$, *ſi loco* x *ponatur* $x - 3$.

Fiet ergo $\omega = -3$; & poſito $y = xx + 3x + 1$, erit $\frac{dy}{dx} = 2x + 3$; $\frac{ddy}{dx^2} = 2$:

unde poſito $x - 3$ loco x functio $x^2 + 3x + 1$ abibit in

$$x^2 + 3x + 1 - \tfrac{3}{1}(xx + 3) + \tfrac{9}{2}.2 = x^2 - 3x + 1.$$

51. Si pro ω ſumatur numerus negativus, reperietur valor, quem functio quaecunque ipſius x induit, dum ipſa quantitas x diminuitur data quantitate ω. Scilicet ſi loco x ponatur $x - \omega$, functio ipſius x quaecunque y accipiet iſtum valorem:

$$y - \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} - \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} - \&c.$$

unde omnes variationes, quas functio y ſubire poteſt, dum quantitas x utrinque variatur, inveniri poterunt. Quodſi autem y fuerit functio rationalis integra ipſius x, quoniam tandem ad eius differentialia evaneſcentia devenitur, valor variatus per expreſſionem finitam exprimetur; ſin autem y non fuerit huiuſmodi functio, valor variatus per ſeriem infinitam exprimetur, cuius propterea ſumma, quoniam ſi ſubſtitutio actu inſtituatur, valor variatus facile aſſignatur, expreſſione finita exhiberi poterit.

52. Quemadmodum autem differentia prima eſt inventa, ita quoque differentiae ſequentes ſimilibus expreſſionibus exhiberi poſſunt. Induat enim x ſucceſſive valores $x + \omega$, $x + 2\omega$, $x + 3\omega$, $x + 4\omega$, &c. atque valores ipſius y reſpondentes indicentur per y^{I}, y^{II}, y^{III}, y^{IV}, &c. ſicuti in initio huius libri poſuimus. Quoniam ergo y^{I}, y^{II}, y^{III}, y^{IV}, &c. ſunt valores, quos y nanciſcitur, ſi loco x ſcribatur reſpective $x + \omega$, $x + 2\omega$, $x + 3\omega$, $x + 4\omega$, &c. per modo demonſtrata iſti ipſius y valores ita exprimentur:

$$y^{I} = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \&c.$$

$$y^{II} = y + \frac{2\omega dy}{dx} + \frac{4\omega^2 ddy}{2dx^2} + \frac{8\omega^3 d^3y}{6dx^3} + \frac{16\omega^4 d^4y}{24dx^4} + \&c.$$

$$y^{III} = y + \frac{3\omega dy}{dx} + \frac{9\omega^2 ddy}{2dx^2} + \frac{27\omega^3 d^3y}{6dx^3} + \frac{81\omega^4 d^4y}{24dx^4} + \&c.$$

$$y^{IV} = y + \frac{4\omega dy}{dx} + \frac{16\omega^2 ddy}{2dx^2} + \frac{64\omega^3 d^3y}{6dx^3} + \frac{256\omega^4 d^4y}{24dx^4} + \&c.$$

&c.

53.

53. Cum igitur, ſi Δy, $\Delta^2 y$, $\Delta^3 y$, $\Delta^4 y$, &c. denotent differentias, primam, ſecundam, tertiam, quartam, &c. ſit:

$$\Delta y = y^{I} - y$$
$$\Delta^2 y = y^{II} - 2y^{I} + y$$
$$\Delta^3 y = y^{III} - 3y^{II} + 3y^{I} - y$$
$$\Delta^4 y = y^{IV} - 4y^{III} + 6y^{II} - 4y^{I} + y$$

iſtae differentiae per differentialia hoc modo exprimentur:

$$\Delta y = \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^4 d^4 y}{24dx^4} + \&c.$$

$$\Delta^2 y = \frac{(2^2 - 2.1)\omega^2 ddy}{2dx^2} + \frac{(2^3 - 2.1)\omega^3 d^3 y}{6dx^3} + \frac{(2^4 - 2.1)\omega^4 d^4 y}{24dx^4} + \&c.$$

$$\Delta^3 y = \frac{(3^3 - 3.2^3 + 3.1)\omega^3 d^3 y}{6dx^3} + \frac{(3^4 - 3.2^4 + 3.1)\omega^4 d^4 y}{24dx^4} + \&c.$$

$$\Delta^4 y = \frac{(4^4 - 4.3^4 + 6.2^4 - 4.1)\omega^4 d^4 y}{24dx^4} + \frac{(4^5 - 4.3^5 + 6.2^5 - 4.1)\omega^5 d^5 y}{120dx^5} + \&c.$$

&c.

54. Quantam utilitatem afferant iſtae differentiarum expreſſiones in doctrina ſerierum & progreſſionum, cum ſponte patet, tum in ſequentibus uberius exponemus. Interim tamen in hoc capite uſum, qui hinc ad ſerierum notitiam immediate redundat, perpendamus. Quamquam vulgo indices terminorum ſeriei cuiuſcunque progreſſionem arithmeticam, cuius differentia eſt unitas, conſtituere aſſumuntur; tamen quo uſus latius pateat, atque applicatio facilius fieri poſſit, differentiam ſtatuamus $= \omega$, ita ut, ſi terminus generalis ſeu is qui indici x reſpondet, fuerit y; ſequentes conveniant indicibus

$$x + \omega,\ x + 2\omega,\ x + 3\omega,\ \&c.$$

Quodſi ergo his indicibus reſpondeant ſequentes ſeriei termini

$$x,\ x + \omega,\ x + 2\omega,\ x + 3\omega,\ x + 4\omega,\ \&c.$$
$$y,\ P,\ Q,\ R,\ S,\ \&c.$$

ſinguli ex y eiuſque differentialibus definientur hoc modo:

$$P = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^4 d^4 y}{24dx^4} + \&c.$$

$$Q = y + \frac{2\omega dy}{dx} + \frac{4\omega^2 ddy}{2dx^2} + \frac{8\omega^3 d^3 y}{6dx^3} + \frac{16\omega^4 d^4 y}{24dx^4} + \&c.$$

$$R = y + \frac{3\omega dy}{dx} + \frac{9\omega^2 ddy}{2dx^2} + \frac{27\omega^3 d^3 y}{6dx^3} + \frac{81\omega^4 d^4 y}{24dx^4} + \&c.$$

$$S = y + \frac{4\omega dy}{dx} + \frac{16\omega^2 ddy}{2dx^2} + \frac{64\omega^3 d^3 y}{6dx^3} + \frac{256\omega^4 d^4 y}{24dx^4} + \&c.$$

&c.

55. Si hae expreſſiones a ſe invicem ſubtrahantur, in differentias non amplius ingredietur y, eritque

$$P - y = \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^4 d^4 y}{24dx^4} + \&c.$$

$$Q - P = \frac{\omega dy}{dx} + \frac{3\omega^2 ddy}{2dx^2} + \frac{7\omega^3 d^3 y}{6dx^3} + \frac{15\omega^4 d^4 y}{24dx^4} + \&c.$$

$$R - Q = \frac{\omega dy}{dx} + \frac{5\omega^2 ddy}{2dx^2} + \frac{19\omega^3 d^3 y}{6dx^3} + \frac{65\omega^4 d^4 y}{24dx^4} + \&c.$$

$$S - R = \frac{\omega dy}{dx} + \frac{7\omega^2 ddy}{2dx^2} + \frac{37\omega^3 d^3 y}{6dx^3} + \frac{175\omega^4 d^4 y}{24dx^4} + \&c.$$

$$T - S = \frac{\omega dy}{dx} + \frac{9\omega^2 ddy}{2dx^2} + \frac{61\omega^3 d^3 y}{6dx^3} + \frac{369\omega^4 d^4 y}{24dx^4} + \&c.$$

&c.

Si hae expreſſiones denuo a ſe invicem ſubtrahantur, etiam differentialia prima ſe deſtruent; eritque

$$Q - 2P + y = \frac{2\omega^2 ddy}{2dx^2} + \frac{6\omega^3 d^3 y}{6dx^3} + \frac{14\omega^4 d^4 y}{24dx^4} + \&c.$$

$$R - 2Q + P = \frac{2\omega^2 ddy}{2dx^2} + \frac{12\omega^3 d^3 y}{6dx^3} + \frac{50\omega^4 d^4 y}{24dx^4} + \&c.$$

$$S - 2R + Q = \frac{2\omega^2 ddy}{2dx^2} + \frac{18\omega^3 d^3 y}{6dx^3} + \frac{110\omega^4 d^4 y}{24dx^4} + \&c.$$

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$$T - 2S + R = \frac{2\omega^2 ddy}{2dx^2} + \frac{24\omega^3 d^3y}{6dx^3} + \frac{194\omega^4 d^4y}{24dx^4} + \&c.$$

&c.

His autem denuo a ſe invicem ſubtractis differentialia quoque ſecunda ex computo egredientur:

$$R - 3Q + 3P - y = \frac{6\omega^3 d^3y}{6dx^3} + \frac{36\omega^4 d^4y}{24dx^4} + \&c.$$

$$S - 3R + 3Q - P = \frac{6\omega^3 d^3y}{6dx^3} + \frac{60\omega^4 d^4y}{24dx^4} + \&c.$$

$$T - 3S + 3R - Q = \frac{6\omega^3 d^3y}{6dx^3} + \frac{84\omega^4 d^4y}{24dx^4} + \&c.$$

ſubtractionem autem ulterius continuando fiet:

$$S - 4R + 6Q - 4P + y = \frac{24\omega^4 d^4y}{24dx^4} + \&c.$$

$$T - 4S + 6R - 4Q + P = \frac{24\omega^4 d^4y}{24dx^4} + \&c.$$ atque

$$T - 5S + 10R - 10Q + 5P - y = \frac{120\omega^5 d^5y}{120dx^5} + \&c.$$

56. Quodſi ergo y fuerit functio rationalis integra ipſius x, quia eius differentialia altiora tandem evaneſcent, hoc modo procedendo tandem ad expreſſiones evaneſcentes pervenietur. Cum igitur iſtae expreſſiones ſint differentiae ipſius y, earum formas & coefficientes diligentius perpendamus:

$$y = y$$

$$\Delta y = \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \frac{\omega^5 d^5y}{120dx^5} + \&c.$$

$$\Delta^2 y = \frac{\omega^2 ddy}{dx^2} + \frac{3\omega^3 d^3y}{3dx^3} + \frac{7\omega^4 d^4y}{3.4dx^4} + \frac{15\omega^5 d^5y}{3.4.5dx^5} + \frac{31\omega^6 d^6y}{3.4.5.6dx^6} + \&c.$$

$$\Delta^3 y = \frac{\omega^3 d^3y}{dx^3} + \frac{6\omega^4 d^4y}{4dx^4} + \frac{25\omega^5 d^5y}{4.5dx^5} + \frac{90\omega^6 d^6y}{4.5.6dx^6} + \frac{301\omega^7 d^7y}{4.5.6.7dx^7} + \&c.$$

$$\Delta^4 y = \frac{\omega^4 d^4 y}{dx^4} + \frac{10\omega^5 d^5 y}{5dx^5} + \frac{65\omega^6 d^6 y}{5.6dx^6} + \frac{350\omega^7 d^7 y}{5.6.7.dx^7} + \&c.$$

$$\Delta^5 y = \frac{\omega^5 d^5 y}{dx^5} + \frac{15\omega^6 d^6 y}{6dx^6} + \frac{140\omega^7 d^7 y}{6.7dx^7} + \frac{1050\omega^8 d^8 y}{6.7.8dx^8} + \&c.$$

$$\Delta^6 y = \frac{\omega^6 d^6 y}{dx^6} + \frac{21\omega^7 d^7 y}{7dx^7} + \frac{266\omega^8 d^8 y}{7.8dx^8} + \frac{2646\omega^9 d^9 y}{7.8.9dx^9} + \&c.$$

&c.

In quibus ſeriebus quemadmodum denominatores procedant, clarum eſt; numeratorum autem coefficientes ita formantur, ut quivis coefficiens numeratoris ſit aggregatum ex ſupra ſtante & praecedente per exponentem differentiae multiplicato. Sic in ſerie differentiam $\Delta^6 y$ exprimente, eſt $2646 = 1050 + 6.266$.

57. Conſideremus quoque ſeriem eandem ſimul retro continuatam, quae continet terminos indicibus

$x-\omega$; $x-2\omega$; $x-3\omega$; &c. reſpondentes:

$x-4\omega$;	$x-3\omega$;	$x-2\omega$;	$x-\omega$;	x;	$x+\omega$;	$x+2\omega$;	$x+3\omega$;	$x+4\omega$;	&c.
s,	r,	q,	p,	y,	P,	Q,	R,	S,	&c.

Cum igitur ſit:

$$p = y - \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} - \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^4 d^4 y}{24dx^4} - \&c.$$

$$q = y - \frac{2\omega dy}{dx} + \frac{4\omega^2 ddy}{2dx^2} - \frac{8\omega^3 d^3 y}{6dx^3} + \frac{16\omega^4 d^4 y}{24dx^4} - \&c.$$

$$r = y - \frac{3\omega dy}{dx} + \frac{9\omega^2 ddy}{2dx^2} - \frac{27\omega^3 d^3 y}{6dx^3} + \frac{81\omega^4 d^4 y}{24dx^4} - \&c.$$

$$s = y - \frac{4\omega dy}{dx} + \frac{16\omega^2 ddy}{2dx^2} - \frac{64\omega^3 d^3 y}{6dx^3} + \frac{256\omega^4 d^4 y}{24dx^4} - \&c.$$

&c.

erit his valoribus a ſuperioribus P, Q, R, S, &c. ſubtrahendis:

$$\frac{P-p}{2} = \frac{\omega dy}{dx} + \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^5 d^5 y}{120dx^5} + \&c.$$

$$\frac{Q-q}{2} = \frac{2\omega dy}{dx} + \frac{8\omega^3 d^3 y}{6dx^3} + \frac{32\omega^5 d^5 y}{120dx^5} + \&c.$$

$$\frac{R-r}{2} = \frac{3\omega dy}{dx} + \frac{27\omega^3 d^3 y}{6dx^3} + \frac{243\omega^5 d^5 y}{120dx^5} + \&c.$$
$$\frac{S-s}{2} = \frac{4\omega dy}{dx} + \frac{64\omega^3 d^3 y}{6dx^3} + \frac{1024\omega^5 d^5 y}{120dx^5} + \&c.$$
&c.

ſin autem termini hi ad ſuperiores addantur, tum, quemadmodum hic differentialia parium ordinum deerant, differentialia imparia ex computo egredientur. Erit enim

$$\frac{P+p}{2} = y + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^4 d^4 y}{24dx^4} + \frac{\omega^6 d^6 y}{720dx^6} + \&c.$$
$$\frac{Q+q}{2} = y + \frac{4\omega^2 ddy}{2dx^2} + \frac{16\omega^4 d^4 y}{24dx^4} + \frac{64\omega^6 d^6 y}{720dx^6} + \&c.$$
$$\frac{R+r}{2} = y + \frac{9\omega^2 ddy}{2dx^2} + \frac{81\omega^4 d^4 y}{24dx^4} + \frac{729\omega^6 d^6 y}{720dx^6} + \&c.$$
$$\frac{S+s}{2} = y + \frac{16\omega^2 ddy}{2dx^2} + \frac{256\omega^4 d^4 y}{24dx^4} + \frac{4096\omega^6 d^6 y}{720dx^6} + \&c.$$
&c.

58. Quoniam termini antecedentes omnes exprimi poſſunt, ſi ii in unam ſummam colligantur, prodibit ſeriei propoſitae terminus ſummatorius. Reſpondeat ſcilicet terminus primus indici $x - n\omega$, eritque ipſe terminus primus $=$

$$y - \frac{n\omega dy}{dx} + \frac{n^2\omega^2 ddy}{2dx^2} - \frac{n^3\omega^3 d^3 y}{6dx^3} + \frac{n^4\omega^4 d^4 y}{24dx^4} \quad \&c.$$

Cum igitur terminus indici x reſpondens ſit $= y$, terminorumque omnium numerus ſit $= n+1$, erit ſumma omnium a primo ad ultimum y incluſive ſumtorum ſeu terminus ſummatorius $=$

$$(n+1)y - \frac{\omega dy}{dx}\,(1+2+3+\ldots+n)$$
$$+\frac{\omega^2 ddy}{2dx^2}\,(1+2^2+3^2+\ldots+n^2)$$

—

$$-\frac{\omega^3 d^3 y}{6dx^3}\,(1+2^3+3^3+\;.\;.\;.\;+n^3)$$
$$+\frac{\omega^4 d^4 y}{24dx^4}\,(1+2^4+3^4+\;.\;.\;.\;+n^4)$$
$$-\frac{\omega^5 d^5 y}{120dx^5}\,(1+2^5+3^5+\;.\;.\;.\;+n^5)$$
&c.

59. Supra autem ſingularum harum ſerierum ſummas exhibuimus, quae ſi hic ſubſtituantur, erit ſumma ſeriei noſtrae propoſitae =

$$(n+1)y-\frac{\omega dy}{dx}\,(\tfrac{1}{2}nn+\tfrac{1}{2}n)$$
$$+\frac{\omega^2 ddy}{2dx^2}\,(\tfrac{1}{3}n^3+\tfrac{1}{2}nn+\tfrac{1}{6}n)$$
$$-\frac{\omega^3 d^3 y}{6dx^3}\,(\tfrac{1}{4}n^4+\tfrac{1}{2}n^3+\tfrac{1}{4}n^2)$$
$$+\frac{\omega^4 d^4 y}{24dx^4}\,(\tfrac{1}{5}n^5+\tfrac{1}{2}n^4+\tfrac{1}{3}n^3-\tfrac{1}{30}n)$$
$$-\frac{\omega^5 d^5 y}{120dx^5}\,(\tfrac{1}{6}n^6+\tfrac{1}{2}n^5+\tfrac{5}{12}n^4-\tfrac{1}{12}n^2)$$
&c.

ubi n dabitur ex indice termini primi, a quo ſumma computatur. Ita ſi ponatur $\omega=1$, & index termini primi ſit $=1$, ſecundi $=2$, & ultimi $=x$, ita ut haec ſeries ſit propoſita:

$$\begin{array}{cccccc} 1, & 2, & 3, & 4, & \ldots\ldots & x \\ a, & b, & c, & d, & \ldots\ldots & y \end{array}$$

erit huius ſeriei ſumma (ob $x-n=1$ & $n=x-1$)

$$=xy-\frac{dy}{dx}\,(\tfrac{1}{2}xx-\tfrac{1}{2}x)$$
$$+\frac{ddy}{2dx^2}\,(\tfrac{1}{3}x^3-\tfrac{1}{2}xx+\tfrac{1}{6}x)$$
—

$$-\frac{d^3y}{6dx^3}\left(\tfrac{1}{4}x^4-\tfrac{1}{2}x^3+\tfrac{1}{4}xx\right)$$
$$+\frac{d^4y}{24dx^4}\left(\tfrac{1}{5}x^5-\tfrac{1}{2}x^4+\tfrac{1}{3}x^3-\tfrac{1}{30}x\right)$$
$$-\frac{d^5y}{120dx^5}\left(\tfrac{1}{6}x^6-\tfrac{1}{2}x^5+\tfrac{5}{12}x^4-\tfrac{1}{12}x^2\right)$$
$$+\frac{d^6y}{720dx^6}\left(\tfrac{1}{7}x^7-\tfrac{1}{2}x^6+\tfrac{1}{2}x^5-\tfrac{1}{6}x^3+\tfrac{1}{42}x\right)$$
&c.

60. Ex hac ſummae expreſſione, quia coefficientes vehementer augentur, ſi x fuerit numerus magnus, parum utilitatis ad doctrinam ſerierum redundat; interim tamen iuvabit aliquas proprietates inde fluentes commemoraſſe. Sit terminus generalis $y=x^n$, atque terminus ſummatorius per Sy ſeu $S.x^n$ indicetur. Qua deſignatione ubique adhibita erit:

$$\tfrac{1}{2}xx-\tfrac{1}{2}x=S.x-x$$
$$\tfrac{1}{3}x^3-\tfrac{1}{2}x^2+\tfrac{1}{6}x=S.x^2-x^2$$
$$\tfrac{1}{4}x^4-\tfrac{1}{2}x^3+\tfrac{1}{4}xx=S.x^3-x^3 \text{ \&c.}$$

Quamobrem ex ſuperiori expreſſione obtinebitur:

$$S.x^n=x^{n+1}-nx^{n-1}S.x+nx^n$$
$$+\frac{n(n-1)}{1.\,2}x^{n-2}S.x^2-\frac{n(n-1)}{1.\,2}x^n$$
$$-\frac{n(n-1)(n-2)}{1.\,2.\,3}x^{n-3}S.x^3+\frac{n(n-1)(n-2)}{1.\,2.\,3}x^n$$
&c.

At cum ſit

$$(1-1)^n=0=1-n+\frac{n(n-1)}{1.\,2}-\frac{n(n-1)(n-2)}{1.\,2.\,3}+\text{\&c.}$$

erit $n-\frac{n(n-1)}{1.\,2}+\frac{n(n-1)(n-2)}{1.\,2.\,3}-\text{\&c.}=1$,

ideoque excepto caſu $n=0$, quo iſta expreſſio fit $=0$.

S.

$$S.x^n = x^{n+1} + x^n - nx^{n-1}S.x + \frac{n(n-1)}{1.\ 2}x^{n-2}S.x^2$$
$$-\frac{n(n-1)(n-2)}{1.\ 2.\ 3}x^{n-3}S.x^3 + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}x^{n-4}S.x^4$$
&c.

61. Quo tam veritas quam vis huius formulae clarius perſpiciatur, evolvamus ſingulos caſus, ſitque primo $n = 1$, eritque: $S.x = x^2 + x - S.x$, ideoque $S.x = \frac{xx + x}{2}$, quemadmodum ſatis conſtat.

Ponamus ergo $n = 2$, & erit:
$$S.x^2 = x^3 + xx - 2xS.x + S.x^2,$$
quae aequatio, cum utrinque termini $S.x^2$ ſe tollant, idem dat, quod praecedens $S.x = \frac{xx+x}{2}$. Si ſit $n = 3$, erit $S.x^3 = x^4 + x^3 - 3x^2S.x + 3xS.x^2 - S.x^3$, ideoque
$$S.x^3 = \tfrac{3}{2}xS.x^2 - \tfrac{3}{2}x^2S.x + \tfrac{1}{2}x^3(x+1),$$
ſi ponatur $n = 4$ prodibit:
$$S.x^4 = x^5 + x^4 - 4x^3S.x + 6x^2S.x^2 - 4xS.x^3 + S.x^4,$$
unde ob $S.x^4$ deſtructum erit:
$$S.x^3 = \tfrac{3}{2}xS.x^2 - x^2S.x + \tfrac{1}{4}x^3(x+1)$$
a cuius triplo, ſi praecedentis duplum ſubtrahatur remanebit:
$$S.x^3 = \tfrac{3}{2}xS.x^2 - \tfrac{1}{4}x^3(x+1).$$
Si ponatur $n = 5$ fiet:
$$S.x^5 = x^6 + x^5 - 5x^4S.x + 10x^3S.x^2 - 10x^2S.x^3 + 5xS.x^4 - S.x^5$$
ſeu
$$S.x^5 = \tfrac{5}{2}xS.x^4 - 5x^2S.x^3 + 5x^3S.x^2 - \tfrac{5}{2}x^4S.x + \tfrac{1}{2}x^5(x+1)$$
atque ex $n = 6$ ſequitur:
$$S.x^6 = x^7 + x^6 - 6x^5S.x + 15x^4S.x^2 - 20x^3S.x^3 + 15x^2S.x^4 - 6xS.x^5 + S.x^6$$
ſeu

ſeu

$$S.x^5 = \tfrac{5}{2} x S.x^4 - \tfrac{10}{3} x^2 S.x^3 + \tfrac{5}{2} x^3 S.x^2 - x^4 S.x + \tfrac{1}{6} x^5 (x+1).$$

62. Ex his ergo generaliter concludimus, ſi fuerit $n = 2m + 1$ fore:

$$S.x^{2n+1} = \frac{2m+1}{2} x S.x^{2m} - \frac{(2m+1)2m}{2.\ 1.\ 2} x^2 S.x^{2m-1} + \frac{(2m+1)2m(2m-1)}{2.\ 1.\ 2.\ 3} x^3 S.x^{2m-2}$$

$$- \ldots \ldots - \frac{(2m+1)}{2} x^{2m} S.x + \tfrac{1}{2} x^{2m+1}(x+1).$$

Sin autem ſit $n = 2m + 2$, quia termini $S.x^{2m+2}$ ſe mutuo deſtruunt, reperietur:

$$S.x^{2m+1} = \frac{2m+1}{2} x S.x^{2m} - \frac{(2m+1)2m}{2.\ 3} x^2 S.x^{2n-1} + \frac{(2m+1)2m(2m-1)}{2.\ 3.\ 4.} x^3 S.x^{2m-2}$$

$$- \ldots \ldots - x^{2m} S.x + \frac{1}{2m+2} x^{2m+1}(x+1).$$

Duplici ergo modo ſummae poteſtatum imparium ex ſummis poteſtatum inferiorum determinari poſſunt: atque ex varia combinatione harum duarum formularum infinitae aliae formari poſſunt.

63. Multo facilius autem ſummae poteſtatum imparium ex antecedentibus definiri poſſunt: atque ad hoc quidem ſufficit ſolam ſummam poteſtatis paris antecedentis noviſſe. Ex ſummis enim poteſtatum ſupra exhibitis conſtat, numerum terminorum ſummas conſtituentium, in paribus tantum poteſtatibus augeri, ita ut ſumma poteſtatis imparis totidem conſtet terminis, quot ſumma poteſtatis paris praecedentis. Sic ſi poteſtatis paris x^{2n} ſumma ſit:

$$S.x^{2n} = \alpha x^{2n+1} + \beta x^{2n} + \gamma x^{2n-1} - \delta x^{2n-3} + \varepsilon x^{2n-5} - \&c.$$

vidimus enim, poſt terminum tertium alternos terminos deficere, ſimulque ſigna alternari; hinc ſumma ſequentis poteſtatis x^{2n+1} invenietur, ſi ſinguli illius termini reſpective multiplicentur per hos numeros:

$$\frac{2n+1}{2n+2} x;\ \frac{2n+1}{2n+1} x;\ \frac{2n+1}{2n} x;\ \frac{2n+1}{2n-1} x;\ \frac{2n+1}{2n-2} x;\ \&c.$$

non omittendo terminos deficientes; eritque ergo

$$S.x^{2n+1} = \frac{2n+1}{2n+2}\alpha x^{2n+2} + \frac{2n+1}{2n+1}\beta x^{2n+1} + \frac{2n+1}{2n}\gamma x^{2n}$$
$$-\frac{2n+1}{2n-1}\delta x^{2n-2} + \frac{2n+1}{2n-4}\varepsilon x^{2n-4} - \frac{2n+1}{2n-6}\zeta x^{2n-6} + \&c.$$

Quodſi ergo conſtet ſumma poteſtatis x^{2n}, ex ea expedite ſumma ſequentis poteſtatis x^{2n+1} formari poterit.

64. Haec ſequentium ſummarum inveſtigatio etiam ad poteſtates pares extenditur; quoniam autem harum ſummae novum terminum recipiunt, hic per iſtam methodum non invenitur, ex natura tamen ipſius ſeriei, qua conſtat, ſi ponatur $x = 1$, ſummam quoque fieri debere $= 1$, ſemper erui poterit. Viciſſim autem ſemper ex ſumma cuiuſvis poteſtatis cognita praecedentium poteſtatum ſummae inveniri poterunt. Si enim fuerit:

$$S.x^n = \alpha x^{n+1} + \beta x^n + \gamma x^{n-1} - \delta x^{n-3} + \varepsilon x^{n-5} - \zeta x^{n-7} + \&c.$$

erit pro poteſtate praecedente:

$$S.x^{n-1} = \frac{n+1}{n}\alpha x^n + \frac{n}{n}\beta x^{n-1} + \frac{(n-1)}{n}\gamma x^{n-2} - \frac{(n-3)}{n}\delta x^{n-4} + \&c.$$

hincque ulterius regredi licet, quouſque libuerit. Notandum autem eſt eſſe perpetuo $\alpha = \frac{1}{n+1}$ & $\beta = \frac{1}{2}$, uti ex formulis iam ſupra datis apparet.

65. Attendenti ſtatim patebit ſummam poteſtatum x^{n-1} prodire, ſi ſumma poteſtatum x^n differentietur, eiuſque differentiale per ndx dividatur; eritque adeo

$d.Sx^n = ndx.Sx^{n-1}$ & quia eſt $d.x^n = nx^{n-1}dx$;

erit $d.Sx^n = S.nx^{n-1}dx = S.d.x^n$;

ex quo intelligitur differentiale ſummae aequari ſummae differentialis: ita in genere ſi ſeriei cuiuſpiam terminus generalis fuerit $= y$ & Sy eius terminus ſummatorius; erit quoque $S.dy = d.Sy$: hoc eſt ſumma differentialium omnium terminorum aequatur differentiali ſummae ipſorum terminorum.

rum. Ratio autem huius aequalitatis facile perſpicitur ex iis, quae ſupra de ſerierum differentiatione attulimus. Cum enim ſit

$$S.x^n = x^n + (x-1)^n + (x-2)^n + (x-3)^n + (x-4)^n + \&c.$$

erit

$$\frac{d.Sx^n}{ndx} = x^{n-1} + (x-1)^{n-1} + (x-2)^{n-1} + (x-3)^{n-1} + \&c. = S.x^{n-1}$$

quae demonſtratio ad omnes alias ſeries patet.

66. Revertamur autem, unde digreſſi ſumus, ad differentias functionum, circa quas adhuc quaedam annotanda ſunt. Quoniam vidimus, ſi y fuerit functio quaecunque ipſius x, atque loco x ubique ponatur $x \pm \omega$, functionem y adepturam eſſe ſequentem valorem:

$$y \pm \frac{\omega dy}{1dx} + \frac{\omega^2 ddy}{1.2dx^2} \pm \frac{\omega^3 d^3y}{1.2.3.dx^3} + \frac{\omega^4 d^4y}{1.2.3.4dx^4} \pm \frac{\omega^5 d^5y}{1.2.3.4.5dx^5} + \&c.$$

haec expreſſio locum habebit, ſive pro ω quantitas quaecunque conſtans accipiatur, ſive etiam variabilis, ab ipſa x pendens. Inventis enim per differentiationem valoribus fractionum $\frac{dy}{dx}$; $\frac{ddy}{dx^2}$; $\frac{d^3y}{dx^3}$; &c. in factoribus ω, ω^2, ω^3, &c. variabilitas non ſpectatur, hincque perinde eſt ſive ω denotet quantitatem conſtantem, ſive variabilem ab x pendentem.

67. Ponamus ergo eſſe $\omega = x$, atque in functione y loco x ſcribi $x - x = 0$. Quamobrem ſi in functione ipſius x quacunque y loco x ubique ſcribatur o, valor functionis erit hic:

$$-\frac{xdy}{1dx} + \frac{x^2 ddy}{1.2dx^2} - \frac{x^3 d^3y}{1.2.3dx^3} + \frac{x^4 d^4y}{1.2.3.4dx^4} - \&c.$$

Haec ergo expreſſio ſemper indicat valorem, quem functio quaecunque y induit, ſi in ea ponatur $x = 0$, cuius veritatem ſequentia exempla illuſtrabunt:

EXEMPLUM I.

Sit $y = xx + ax + ab$, cuius valor, ſi ponatur $x = 0$, quaeratur, quem quidem conſtat fore $= ab$

Oo 2 Cum

Cum ſit $y = xx + ax + ab$ erit

$$\frac{dy}{1dx} = 2x + a \;;\quad \frac{ddy}{1.2dx^2} = 1$$

ideoque prodibit valor quaeſitus

$$= xx + ax + ab - x(2x + a) + xx.\ 1 = ab.$$

EXEMPLUM II.

Sit $y = x^3 - 2x + 3$, cuius valor, poſito $x = 0$, quaeratur, quem conſtat fore $= 3$.

Cum ſit $y = x^3 - 2x + 3$ erit

$$\frac{dy}{dx} = 3xx - 2 \;;\quad \frac{ddy}{1.2dx^2} = 3x \;;\quad \frac{d^3y}{1.2.3dx^3} = 1$$

obtinebitur valor quaeſitus

$$= x^3 - 2x + 3 - x(3xx - 2) + xx.\ 3x - x^3.\ 1 = 3.$$

EXEMPLUM III.

Sit $y = \frac{x}{1-x}$, cuius valor poſito $x = 0$, quaeritur, quem conſtat fore $= 0$.

Cum ſit $y = \frac{x}{1-x}$; erit $\frac{dy}{dx} = \frac{1}{(1-x)^2}$;

$$\frac{ddy}{1.2dx^2} = \frac{1}{(1-x)^3};\ \frac{d^3y}{1.2.3dx^3} = \frac{1}{(1-x)^4};\ \text{\&c.}$$

Hinc erit valor quaeſitus

$$= \frac{x}{1-x} - \frac{x}{(1-x)^2} + \frac{xx}{(1-x)^3} - \frac{x^3}{(1-x)^4} + \frac{x^4}{(1-x)^5} - \text{\&c.}$$

huiuſque ergo ſeriei valor eſt $= 0$.

Quod etiam hinc patet, quod haec ſeries primo termino truncata $\frac{x}{(1-)^2} - \frac{xx}{(1-x)^3} + \frac{x^3}{(1-x)^4} - \text{\&c.}$ ſit ſeries geometrica, eiuſque ſumma $= \frac{x}{(1-x)^2 + x(1-x)} = \frac{x}{1-x}$; unde valor inventus erit $= \frac{x}{1-x} - \frac{x}{1-x} = 0$.

EXEMPLUM IV.

Sit $y = e^x$, denotante e numerum, cuius logarithmus hyperbolicus eſt unitas, & quaeratur valor ipſius y ſi ponatur $x = 0$, quem quidem conſtat fore $= 1$.

Cum ſit $y = e^x$; erit $\frac{dy}{dx} = e^x$; $\frac{ddy}{dx^2} = e^x$; &c.

ideoque valor quaeſitus erit

$$= e^x - \frac{e^x x}{1} + \frac{e^x xx}{1.2} - \frac{e^x x^3}{1.2.3} + \frac{e^x x^4}{1.2.3.4} - \&c.$$

$$= e^x \left(1 - \frac{x}{1} + \frac{xx}{1.2} - \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} - \&c.\right)$$

At ſupra vidimus ſeriem $1 - \frac{x}{1} + \frac{xx}{1.2} - \frac{x^3}{1.2.3} + \&c.$ exprimere valorem e^{-x}, erit ergo valor quaeſitus utique

$$= e^x . e^{-x} = \frac{e^x}{e^x} = 1.$$

EXEMPLUM V.

Sit $y = \text{ſin}\, x$, atque poſito $x = 0$ manifeſtum eſt fore $y = 0$, id quod etiam formula generalis indicabit.

Cum enim ſit $y = \text{ſin}\, x$; erit $\frac{dy}{dx} = \text{coſ}\, x$;

$\frac{ddy}{dx^2} = -\text{ſin}\, x$; $\frac{d^3 y}{dx^3} = -\text{coſ}\, x$; $\frac{d^4 y}{dx^4} = \text{ſin}\, x$; &c.

erit poſito $x = 0$ valor ipſius y hic:

$$\text{ſin}\, x - \frac{x}{1}\text{coſ}\, x - \frac{xx}{1.2}\text{ſin}\, x + \frac{x^3}{1.2.3}\text{coſ}\, x + \frac{x^4}{1.2.3.4}\text{ſin}\, x - \&c.$$

qui eſt $= \text{ſin}\, x \left(1 - \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3 \ldots 6} + \&c.\right)$

$$- \text{coſ}\, x \left(\frac{x}{1} - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3 \ldots 7} + \&c.\right)$$

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harum autem ſerierum ſuperior exprimit $\cos x$, inferior autem $\sin x$, unde valor quaeſitus erit

$$= \sin x . \cos x - \cos x . \sin x = 0.$$

68. Hinc igitur viciſſim cognoſcimus, ſi y eiuſmodi fuerit functio ipſius x, ut ipſa evaneſcat, poſito $x = 0$, tum fore

$$y - \frac{xdy}{1dx} + \frac{xxddy}{1.2dx^2} - \frac{x^3d^3y}{1.2.3dx^3} + \frac{x^4d^4y}{1.2.3.4dx^4} - \&c. = 0.$$

Unde haec eſt aequatio generalis omnium omnino functionum ipſius x, quae dum fit $x = 0$, ſimul ipſae evaneſcunt. Et hancobrem iſta aequatio ita eſt comparata, ut quaecunque functio ipſius x, dummodo ea evaneſcat evaneſcente x, loco y ſubſtituatur, aequationi perpetuo ſatisfiat. Quodſi vero y eiuſmodi fuerit functio ipſius x, quae poſito $x = 0$, recipiat valorem datum $= A$, tum erit:

$$y - \frac{xdy}{1dx} + \frac{x^2ddy}{1.2dx^2} - \frac{x^3d^3y}{1.2.3dx^3} + \frac{x^4d^4y}{1.2.3.4dx^4} - \&c. = A.$$

in qua aequatione omnes continentur functiones ipſius x, quae poſito $x = 0$, abeunt in A.

69. Si loco x ſcribatur $2x$, ſeu $x+x$, functio quaecunque ipſius x, quae deſignetur per y hunc induet valorem

$$y + \frac{xdy}{1dx} + \frac{x^2ddy}{1.2dx^2} + \frac{x^3d^3y}{1.2.3dx^3} + \frac{x^4d^4y}{1.2.3.4dx^4} + \&c.$$

Atque ſi loco x ſcribamus nx, hoc eſt $x + (n-1)x$ functio y accipiet valorem ſequentem:

$$y + \frac{(n-1)xdy}{1dx} + \frac{(n-1)^2xxddy}{1.2dx^2} + \frac{(n-1)^3x^3d^3y}{1.2.3dx^3} + \&c.$$

Sin autem generaliter pro x ſcribamus t, functio quaecunque y ipſius x, transmutabitur ob $t = x + t - x$ in formam ſequentem:

$$y + \frac{(t-x)dy}{1dx} + \frac{(t-x)^2ddy}{1.2dx^2} + \frac{(t-x)^3d^3y}{1.2.3dx^3} + \&c.$$

Si

Si igitur v fuerit talis functio ipſius t, qualis y eſt ipſius x, quia v ex y naſcitur, ponendo t loco x, erit:

$$v = y + \frac{(t-x)dy}{1dx} + \frac{(t-x)^2 ddy}{1.2dx^2} + \frac{(t-x)^3 d^3y}{1.2.3dx^3} + \&c.$$

cuius veritas quibuſcunque exemplis comprobari poteſt.

EXEMPLUM.

Sit enim $y = xx - x$: manifeſtum eſt poſito t loco x fore $v = tt - t$, quod idem expreſſio inventa declarabit. Nam ob

$$y = xx - x; \text{ erit } \frac{dy}{dx} = 2x - 1; \quad \& \quad \frac{ddy}{2dx^2} = 1;$$

unde fiet

$$v = xx - x + (t-x)(2x-1) + (t-x)^2 =$$
$$xx - x + 2tx - 2xx - t + x + tt - 2tx + xx = tt - t.$$

Si itaque y fuerit eiuſmodi functio ipſius x, quae poſito $x = a$ abeat in A; ob $t = a$ & $v = A$ fiet

$$A = y + \frac{(a-x)dy}{1dx} + \frac{(a-x)^2 ddy}{1.2dx^2} + \frac{(a-x)^3 d^3y}{1.2.3dx^3} + \&c.$$

huicque ergo aequationi omnes functiones ipſius x, quae facto $x = a$ abeunt in A, ſatisfaciunt.

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CAPUT IV.

DE CONVERSIONE FUNCTIONUM IN SERIES.

70.

In Capite ſuperiori iam ex parte oſtenſus eſt uſus, quem expreſſiones generales ibi pro differentiis finitis inventae habent in inveſtigatione ſerierum, quae valorem cuiuſque functionis ipſius x exhibeant. Si enim y fuerit functio data ipſius x, eius valor quem induit poſito $x = 0$, erit cognitus; hicque ſi ponatur $= A$, erit uti invenimus:

$$y - \frac{xdy}{dx} + \frac{x^2 ddy}{1.2dx^2} - \frac{x^3 d^3 y}{1.2.3dx^3} + \frac{x^4 d^4 y}{1.2.3.4dx^4} - \&c. = A.$$

Hinc ergo non ſolum habemus ſeriem plerumque in infinitum excurrentem, cuius ſumma aequetur quantitati conſtanti A, etiamſi in ſingulis terminis inſit quantitas variabilis x, ſed etiam ipſam functionem y per ſeriem exprimere poterimus, erit enim:

$$y = A + \frac{xdy}{dx} - \frac{xxddy}{1.2dx^2} + \frac{x^3 d^3 y}{1.2.3dx^3} - \frac{x^4 d^4 y}{1.2.3.4dx^4} + \&c.$$

cuius exempla iam aliquot ſunt allata.

71. Quo autem haec inveſtigatio latius pateat, ponamus functionem y abire in z, ſi loco x ubique ſcribatur $x + \omega$, ita ut z talis ſit functio ipſius $x + \omega$, qualis y eſt ipſius x, atque oſtendimus fore:

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{1.2dx^2} + \frac{\omega^3 d^3 y}{1.2.3dx^3} + \frac{\omega^4 d^4 y}{1.2.3.4dx^4} + \&c.$$

Cum igitur huius ſeriei ſinguli termini per continuam ipſius y differentiationem ponendo dx conſtans inveniri, ſimulque valor ipſius z per ſubſtitutionem $x + \omega$ in locum ipſius x actu exhiberi queat; hoc modo perpetuo obtinebitur ſeries valori ipſi-

ipſius z aequalis, quae ſi ω fuerit quantitas vehementer parva, maxime convergit, atque non admodum multis terminis capiendis valorem ipſius z proxime verum praebebit. Ex quo huius formulae in negotio approximationum uberrimus erit uſus.

72. Ut igitur in inſigni huius formulae uſu oſtendendo ordine procedamus, ſubſtituamus primo in locum ipſius y functiones ipſius x algebraicas. Ac primo quidem ſit $y = x^n$; eritque ſi $x + \omega$ loco x ponatur $z = (x + \omega)^n$. Cum igitur ſit:

$$\frac{dy}{dx} = nx^{n-1} \quad ; \quad \frac{ddy}{dx^2} = n(n-1)x^{n-2}$$

$$\frac{d^3y}{dx^3} = n(n-1)(n-2)x^{n-3} \; ; \quad \frac{d^4y}{dx^4} = n(n-1)(n-2)(n-3)x^{n-4}$$

&c.

his valoribus ſubſtitutis fiet:

$$(x+\omega)^n = x^n + \frac{n}{1}x^{n-1}\omega + \frac{n(n-1)}{1.2}x^{n-2}\omega^2 + \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}\omega^3 + \&c.$$

quae eſt notiſſima expreſſio Neutoniana, qua poteſtas binomii $(x+\omega)^n$ in ſeriem convertitur. Huiuſque ſeriei terminorum numerus ſemper eſt finitus, ſi n fuerit numerus integer affirmativus.

73. Poterimus hinc quoque progreſſionem invenire, quae valorem poteſtatis binomii ita exprimat, ut ea abrumpatur, quoties exponens poteſtatis fuerit numerus negativus. Statuamus enim

$$\omega = \frac{-ux}{x+u}; \quad \text{erit} \quad z = (x+\omega)^n = \left(\frac{xx}{x+u}\right)^n$$

ideoque habebitur:

$$\frac{x^{2n}}{(x+u)^n} = x^n - \frac{nx^n u}{1(x+u)} + \frac{n(n-1)x^n u^2}{1.2(x+u)^2} - \frac{n(n-1)(n-2)x^n u^3}{1.2.3(x+u)^3} + \&c.$$

dividatur ubique per x^{2n}, eritque

$$(x+u)^{-n} = x^{-n} - \frac{nx^{-n}u}{1(x+u)} + \frac{n(n-1)x^{-n}u^2}{1.2(x+u)^2} - \frac{n(n-1)(n-2)x^{-n}u^3}{1.2.3(x+u)^3} + \&c.$$

Ponatur nunc $-n = m$; prodibitque

$$(x+u)^m = x^m + \frac{mx^m u}{1(x+u)} + \frac{m(m+1)x^m u^2}{1.2(x+u)^2} + \frac{m(m+1)(m+2)x^m u^3}{1.2.3(x+u)^3} + \&c.$$

quae ſeries, quoties m eſt numerus integer negativus, finito terminorum numero conſtabit. Haec igitur ſeries aequalis eſt primum inventae, ſi pro ω & n ſcribantur u & m; erit enim inde

$$(x+u)^m = x^m + \frac{mx^{m-1}u}{1} + \frac{m(m-1)x^{m-2}u^2}{1.\ 2} + \frac{m(m-1)(m-2)x^{m-3}u^3}{1.\ 2.\ 3} + \&c.$$

74. Haec eadem ſeries quoque deduci poteſt ex expreſſione initio §. 70. data. Cum enim, ſi poſito $x = 0$, abeat y in A ſit:

$$y - \frac{xdy}{dx} + \frac{xxddy}{1.2dx^2} - \frac{x^3d^3y}{1.2.3dx^3} + \frac{x^4d^4y}{1.2.3.4dx^4} - \&c. = A,$$

ponatur $y = (x+a)^n$; eritque $A = a^n$; & ob

$$\frac{dy}{dx} = n(x+a)^{n-1}\ ; \quad \frac{ddy}{dx^2} = n(n-1)(x+a)^{n-2};$$

$$\frac{d^3y}{dx^3} = n(n-1)(n-2)(x+a)^{n-3};\ \&c. \text{ fiet}$$

$$(x+a)^n - \frac{n}{1}x(x+a)^{n-1} + \frac{n(n-1)}{1.\ 2}x^2(x+a)^{n-2} - \&c. = a^n$$

dividatur per $a^n(x+a)^n$, atque prodibit:

$$(x+a)^{-n} = a^{-n} - \frac{na^{-n}x}{1(x+a)} + \frac{n(n-1)a^{-n}x^2}{1.2(x+a)^2} - \&c.$$

quae poſitis reſpective u, x & $-m$ pro x, a & n orietur ſeries ante inventa.

75. Si pro m ſtatuantur numeri fracti, ambae ſeries in infinitum excurrent, interim tamen ſi u prae x fuerit quantitas valde parva, vehementer ad verum valorem convergent. Sit igitur $m = \frac{\mu}{\nu}$; & $x = a^\nu$, erit ex ſerie primum inventa:

$$(a^\nu+u)^{\frac{\mu}{\nu}}=a^\mu\left(1+\frac{\mu u}{\nu.a^\nu}+\frac{\mu(\mu-\nu)}{\nu.\ 2\nu}\cdot\frac{uu}{a^{2\nu}}+\frac{\mu(\mu-\nu)(\mu-2\nu)}{\nu.\ 2\nu.\ 3\nu}\cdot\frac{u^3}{a^{3\nu}}+\&c.\right)$$

Series autem poſterius inventa dabit:

$$(a^\nu+u)^{\frac{\mu}{\nu}}=a^\mu\left(1+\frac{\mu u}{\nu(a^\nu+u)}+\frac{\mu(\mu+\nu)u^2}{\nu.2\nu(a^\nu+u)^2}+\frac{\mu(\mu+\nu)(\mu+2\nu)u^3}{\nu.\ 2\nu.\ 3\nu\,(a^\nu+u)^3}+\&c.\right)$$

Haec autem poſterior ſeries magis convergit quam prior, cum eius termini etiam decreſcant, ſi fuerit $u>a^\nu$, quo caſu tamen prior ſeries divergit.

Si igitur ſit $\mu=1$, $\nu=2$, erit

$$\sqrt{(a^2+u)}=a\left(1+\frac{1\ u}{2(a^2+u)}+\frac{1.\ 3.\ u^2}{2.4(a^2+u)^2}+\frac{1.\ 3.\ 5\ u^3}{2.4.6(a^2+u)^3}+\&c.\right)$$

ſimili modo pro ν ponendo numeros 3, 4, 5 &c.
manente $\mu=1$, erit:

$$\sqrt[3]{(a^3+u)}=a\left(1+\frac{1\ u}{3(a^3+u)}+\frac{1.\ 4.\ u^2}{3.6(a^3+u)^2}+\frac{1.\ 4.\ 7\ u^3}{3.6.9(a^4+u)^3}+\&c.\right)$$

$$\sqrt[4]{(a^4+u)}=a\left(1+\frac{1\ u}{4(a^4+u)}+\frac{1.\ 5.\ u^2}{4.8(a^4+u)^2}+\frac{1.\ 5.\ 9\ u^3}{4.8.12(a^4+u)^3}+\&c.\right)$$

$$\sqrt[5]{(a^5+u)}=a\left(1+\frac{1\ u}{5(a^5+u)}+\frac{1.\ 6\ u^2}{5.10(a^5+u)^2}+\frac{1.\ 6.\ 11\ u^3}{5.10.15(a^5+u)^3}+\&c.\right)$$

&c.

76. Ex his ergo formulis facile cuiuſque numeri propoſiti radix cuiuſvis poteſtatis inveniri poterit. Propoſito enim numero c quaeratur poteſtas ei proxima, ſive major ſive minor: priori caſu u fiet numerus negativus, poſteriori affirmativus. Quod ſi vero ſeries reſultans non ſatis convergere videatur, multiplicetur numerus c per quampiam poteſtatem puta per f^ν, ſi radix dignitatis ν extrahi debeat, & quaeratur numeri $f^\nu c$ radix, quae per f diviſa dabit radicem numeri c quaeſitam. Quo major autem accipitur numerus f, eo magis ſeries converget; idque imprimis, ſi quaepiam ſimilis poteſtas a^ν non multum ab $f^\nu c$ diſcrepet.

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EXEMPLUM I.

Quaeratur radix quadrata ex numero 2.

Si ſine ulteriori praeparatione ponatur $a = 1$ & $u = 1$ fiet

$$\sqrt{2} = 1 + \frac{1}{2.2} + \frac{1.3}{2.4.2^2} + \frac{1.3.5}{2.4.6.2^3} + \&c.$$

quae etſi iam ſatis convergit, tamen praeſtabit numerum 2 ante per quadratum quodpiam uti 25 multiplicare, ut productum 50 ab alio quadrato 49 minime diſcrepet. Hancob rem quaeratur radix quadrata ex 50, quae per 5 diviſa dabit $\sqrt{2}$. Erit autem tum $a = 7$ & $u = 1$, unde fiet:

$$\sqrt{50} = 5\sqrt{2} = 7\left(1 + \frac{1}{2.50} + \frac{1.3}{2.4.50^2} + \frac{1.3.5}{2.4.6.50^3} + \&c.\right)$$

ſeu

$$\sqrt{2} = \tfrac{7}{5}\left(1 + \frac{1}{100} + \frac{1.3}{100.200} + \frac{1.3.5}{100.200.300} + \&c.\right)$$

quae ad computum in fractionibus decimalibus inſtituendum eſt aptiſſima.

Erit enim

$\frac{7}{5} =$	1,400000000000
$\frac{7}{5}\cdot\frac{1}{100} =$	140000000000
$\frac{7}{5}\cdot\frac{1}{100}\cdot\frac{3}{200} =$	2100000000
$\frac{7}{5}\cdot\frac{1}{100}\cdot\frac{3}{200}\cdot\frac{5}{300} =$	35000000
praec. in $\frac{7}{400} =$	612500
praec. in $\frac{9}{500} =$	11025
praec. in $\frac{11}{600} =$	202
praec. in $\frac{13}{700} =$	3

Ergo $\sqrt{2} = 1,4142135623730$

EXEMPLUM II.

Quaeratur radix cubica ex 3.

Multiplicetur 3 per cubum 8, & quaeratur radix cubica ex 24, erit enim $\sqrt[3]{24} = 2\sqrt[3]{3}$. Ponatur ergo $a = 3$ & $u = -3$, eritque

$$\sqrt[3]{24} = 3\left(1 - \frac{1\cdot3}{3\cdot24} + \frac{1\cdot4\cdot3^2}{3\cdot6\cdot24^2} - \frac{1\cdot4\cdot7\cdot3^3}{3\cdot6\cdot9\cdot24^3} + \&c.\right) \quad \&$$

$$\sqrt[3]{3} = \tfrac{1}{2}\left(1 - \frac{1}{3\cdot8} + \frac{1\cdot4}{3\cdot6\cdot8^2} - \frac{1\cdot4\cdot7}{3\cdot6\cdot9\cdot8^3} + \&c.\right) \quad \text{ſeu}$$

$$\sqrt[3]{3} = \tfrac{1}{2}\left(1 - \frac{1}{24} + \frac{1}{24}\cdot\frac{4}{48} - \frac{1}{24}\cdot\frac{4}{48}\cdot\frac{7}{72} + \&c.\right)$$

quae ſeries iam vehementer convergit, cum quilibet terminus pluſquam octies minor ſit praecedente. Sin autem 3 multiplicetur per cubum 729 fiet 2187, &

$$\sqrt[3]{2187} = \sqrt[3]{(13^3 - 10)} = 9\sqrt[3]{3}.$$

Erit ergo ob $a = 13$ & $b = -10$

$$\sqrt[3]{3} = \tfrac{13}{9}\left(1 - \frac{1\cdot10}{3\cdot2187} + \frac{1\cdot4\cdot10^2}{3\cdot6\cdot2187^2} - \frac{1\cdot4\cdot7\cdot10^3}{3\cdot6\cdot9\cdot2187^3} + \&c.\right)$$

cuius quivis terminus pluſquam ducenties minor eſt quam praeced.

77. Evolutio binomii poteſtatis tam late patet, ut omnes functiones algebraicae in ea comprehendi queant. Si enim ver. gr. quaeratur valor huius functionis $\sqrt{(a + 2bx + cxx)}$ per ſeriem expreſſus, hoc per praecedentes formulas, duos terminos tanquam unum conſiderando fieri poterit. Deinde vero haec explicatio fieri poterit ope expreſſionis primum traditae: nam ſi ponatur $\sqrt{(a + 2bx + cxx)} = y$, quia poſito $x = 0$ fit $y = \sqrt{a}$, erit $A = \sqrt{a}$, & cum differentialia ipſius y ita ſe habeant:

$$\frac{dy}{dx} = \frac{b + cx}{\sqrt{(a + 2bx + cxx)}}$$

$$\frac{ddy}{dx^2} = \frac{ac - bb}{(a + 2bx + cxx)^{\frac{3}{2}}}$$

$$\frac{d^3y}{dx^3} = +\frac{3(bb - ac)(b + cx)}{(a + 2bx + cxx)^{\frac{5}{2}}}$$

$$\frac{d^4y}{dx^4} = \frac{3(bb - ac)(ac - 5bb - 8bcx - 4ccxx)}{(a + 2bx + cxx)^{\frac{7}{2}}}$$

&c.

Ex

Ex his ergo obtinebitur:

$$\sqrt{(a+2bx+cxx)} - \frac{(b+cx)x}{\sqrt{(a+2bx+cxx)}} - \frac{(bb-ac)xx}{2(a+2bx+cxx)^{\frac{3}{2}}} - \frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^{\frac{5}{2}}}$$
$$- \frac{(bb-ac)(5bb-ac+8bcx+4ccxx)x^4}{8(a+2bx+cxx)^{\frac{7}{2}}} - \&c. = \sqrt{a}.$$

Quodſi ergo ubique per $\sqrt{(a+2bx+cxx)}$ multiplicetur ſeries fiet rationalis, eritque

$$\sqrt{a(a+2bx+cxx)} = a+2bx+cxx-(b+cx)x - \frac{(bb-ac)xx}{2(a+2bx+cxx)} -$$
$$\frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^2} - \frac{(bb-ac)(5bb-ac+8bcx+4ccxx)x^4}{8(a+2bx+cxx)^3} - \&c.$$

ſive

$$\sqrt{(a+2bx+cxx)} = \sqrt{a} + \frac{bx}{\sqrt{a}} - \frac{(bb-ac)xx}{2(a+2bx+cxx)\sqrt{a}} - \frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^2\sqrt{a}} \&c.$$

78. Tranſeamus ergo ad functiones tranſcendentes, quas loco y ſubſtituamus. Sit itaque primum $y = lx$, ac poſito $x+\omega$ loco x fiet $z = l(x+\omega)$. Sint autem hi logarithmi quicunque, qui ad hiperbolicos rationem teneant $n : 1$, eritque pro logarithmis hyperbolicis $n = 1$ & pro tabularibus erit $n = 0,43429448190032$. Hinc differentialia ipſius $y = lx$ erunt: $\frac{dy}{dx} = \frac{n}{x}$; $\frac{ddy}{dx^2} = -\frac{n}{x^2}$; $\frac{d^3y}{dx^3} = \frac{2n}{x^3}$; &c. ex quibus conficitur: $l(x+\omega) = lx + \frac{n\omega}{x} - \frac{n\omega^2}{2x^2} + \frac{n\omega^3}{3x^3} - \frac{n\omega^4}{4x^4} + \&c.$

Simili modo ſi ω ſtatuatur negativum, erit:

$$l(x-\omega) = lx - \frac{n\omega}{x} - \frac{n\omega^2}{2x^2} - \frac{n\omega^3}{3x^3} - \frac{n\omega^4}{4x^4} - \&c.$$

Quodſi ergo haec ſeries a priori ſubtrahatur, fiet

$$l\frac{x+\omega}{x-\omega} = 2n\left(\frac{\omega}{x} + \frac{\omega^3}{3x^3} + \frac{\omega^5}{5x^5} + \frac{\omega^7}{7x^7} + \&c.\right)$$

79. Si in ſerie primum inventa:

$$l(x+\omega)=lx+\frac{n\omega}{x}-\frac{n\omega^2}{2x^2}+\frac{n\omega^3}{3x^3}-\frac{n\omega^4}{4x^4}+\&c.$$

ponatur $\omega=\frac{xx}{u-x}$; erit $x+\omega=\frac{ux}{u-x}$; &

$$l(x+\omega)=lu+lx-l(u-x)=lx+\frac{nx}{u-x}-\frac{nxx}{2(u-x)^2}+\&c.$$

atque

$$l(u-x)=lu-\frac{nx}{u-x}+\frac{nxx}{2(u-x)^2}-\frac{nx^3}{3(u-x)^3}+\&c.$$

ſumtoque x negativo habebitur:

$$l(u+x)=lu+\frac{nx}{u+x}+\frac{nxx}{2(u+x)^2}+\frac{nx^3}{3(u+x)^3}+\frac{nx^4}{4(u+x)^4}.$$

Harum ergo ſerierum ope logarithmi expedite inveniri poterunt, ſi quidem ſeries valde convergant. Huiuſmodi autem erunt ſequentes, quae ex inventis facile deducuntur:

$$l(x+1)=lx+n\left(\frac{1}{x}-\frac{1}{2xx}+\frac{1}{3x^3}-\frac{1}{4x^4}+\&c.\right)$$

$$l(x-1)=lx-n\left(\frac{1}{x}+\frac{1}{2xx}+\frac{1}{3x^3}+\frac{1}{4x^4}+\&c.\right)$$

quae duae ſeries, cum tantum ſignis a ſe invicem diſcrepent, ſi ad calculum revocentur, ex logarithmo numeri x cognito, eadem opera logarithmi amborum numerorum $x+1$ & $x-1$ reperientur. Deinde ex reliquis ſeriebus erit:

$$l(x+1)=l(x-1)+2n\left(\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\frac{1}{7x^7}+\&c.\right)$$

$$l(x-1)=lx-n\left(\frac{1}{x-1}-\frac{1}{2(x-1)^2}+\frac{1}{3(x-1)^3}-\frac{1}{4(x-1)^4}+\&c.\right)$$

$$l(x+1)=lx+n\left(\frac{1}{x+1}+\frac{1}{2(x+1)^2}+\frac{1}{3(x+1)^3}+\frac{1}{4(x+1)^4}+\&c.\right)$$

80. Ex dato ergo logarithmo numeri x, logarithmi nu-

numerorum contiguorum $x+1$ & $x-1$ facile inveniri poterunt; quin etiam ex logarithmo numeri $x-1$ logarithmus numeri binario maioris & viciſſim eruetur. Quod quamvis in Introductione uberius ſit oſtenſum, tamen hic quaedam exempla adiungemus.

EXEMPLUM I.

Ex dato numeri 10 *logarithmo hyperbolico*, *qui eſt* 2,302585092994o, *logarithmos hyperbolicos numerorum* 11 & 9 *invenire.*

Quoniam haec quaeſtio logarithmos hyperbolicos ſpectat, erit $n=1$; ideoque habebuntur hae ſeries:

$$l11 = l10 + \frac{1}{10} - \frac{1}{2.\,10^2} + \frac{1}{3.\,10^3} - \frac{1}{4.\,10^4} + \frac{1}{5.\,10^5} - \&c.$$

$$l9 = l10 - \frac{1}{10} - \frac{1}{2.\,10^2} - \frac{1}{3.\,10^3} - \frac{1}{4.\,10^4} - \frac{1}{5.\,10^5} - \&c.$$

Ad quarum ſerierum ſummas inveniendas, colligantur termini pares & impares ſeorſim, eritque

$\frac{1}{10} = 0{,}1000000000000$	$\frac{1}{2.\,10^2} = 0{,}0050000000000$
$\frac{1}{3.\,10^3} = 0{,}0003333333333$	$\frac{1}{4.\,10^4} = 0{,}0000250000000$
$\frac{1}{5.\,10^5} = 0{,}0000020000000$	$\frac{1}{6.\,10^6} = 0{,}0000001666666$
$\frac{1}{7.\,10^7} = 0{,}0000000142857$	$\frac{1}{8.\,10^8} = 0{,}0000000012500$
$\frac{1}{9.\,10^9} = 0{,}0000000001111$	$\frac{1}{10.10^{10}} = 0{,}0000000000100$
$\frac{1}{11.10^{11}} = 0{,}0000000000009$	$\frac{1}{12.10^{12}} = 0{,}0000000000001$
ſumma $= 0{,}1003353477310$	ſumma $= 0{,}0050251679267$

Sum-

Summa utriuſque erit . . .	0,10536051565577
Differentia ambarum erit	0,0953101798043
Iam eſt	$l10 = 2,302585092994\overline{0}$
Ergo erit	$l11 = 2,397895272798\overline{3}$
&	$l9 = 2,1972245773363$
Hinc porro erit	$l3 = 1,0986122886681$
&	$l99 = 4,5951198501346$

EXEMPLUM II.

Ex logarithmo hyperbolico numeri 99 nunc invento invenire logarithmum numeri 101.

Adhibeatur ad hoc ſeries ſupra inventa:

$$l(x+1) = l(x-1) + \frac{2}{x} + \frac{2}{3x^3} + \frac{2}{5x^5} + \frac{2}{7x^7} + \&c.$$

in qua fiat $x = 100$; eritque:

$$l101 = l99 + \frac{2}{100} + \frac{2}{3.\ 100^3} + \frac{2}{5.\ 100^5} + \frac{2}{7.\ 100^7} + \&c.$$

cuius ſeriei ſumma ex his quatuor terminis colligitur $= 0,0200006667066$, quae ad $l99$ addita dabit $l101 = 4,6151205168412$.

EXEMPLUM III.

Ex dato logarithmo tabulari numeri 10, qui eſt $= 1$, *invenire logarithmos numerorum* 11 & 9.

Quoniam hic logarithmos communes tabulares quaerimus, erit $n = 0,43429448190325$, poſito ergo $x = 10$ erit:

$$l11 = l10 + \frac{n}{10} + \frac{n}{2.\ 10^2} + \frac{n}{3.\ 10^3} + \frac{n}{4.\ 10^4} + \&c.$$

$$l9 = l10 - \frac{n}{10} - \frac{n}{2.\ 10^2} - \frac{n}{3.\ 10^3} - \frac{n}{4.\ 10^4} - \&c.$$

Colligantur termini pares & impares ſeorſim:

$\frac{n}{10} = 0{,}0434294481903$	$\frac{n}{2.10^2} = 0{,}0021714724095$
$\frac{n}{3.10^3} = 0{,}0001447648273$	$\frac{n}{4.10^4} = 0{,}0000108573620$
$\frac{n}{5.10^5} = 0{,}0000008685889$	$\frac{n}{6.10^6} = 0{,}0000000723824$
$\frac{n}{7.10^7} = 0{,}0000000062042$	$\frac{n}{8.10^8} = 0{,}0000000005428$
$\frac{n}{9.10^9} = 0{,}0000000000482$	$\frac{n}{10.10^{10}} = 0{,}0000000000043$
$\frac{n}{11.10^{11}} = 0{,}0000000000003$	$\frac{n}{12.10^{12}} = 0{,}0000000000000$
ſumma $= 0{,}0435750878593$	ſumma $= 0{,}0021824027010$

Aggregatum ambarum eſt $= 0{,}0457574905603$
Differentia earum eſt $= 0{,}0413926851583$
Cum ergo ſit $l10 = 1{,}0000000000000$
Erit $l11 = 1{,}0413926851583$
& $l9 = 0{,}9542425094396$
Hinc $l3 = 0{,}4771212547198$
& $l99 = 1{,}9956351945979$

EXEMPLUM IV.

Ex logarithmo tabulari numeri 99 hic invento invenire logarithmum tabularem numeri 101.

Adhibendo hic eandem ſeriem, qua in Exemplo ſecundo uſi ſumus, habebimus:

$$l101 = l99 + 2n\left(\frac{1}{100} + \frac{1}{3.100^3} + \frac{1}{5.100^5} + \&c.\right)$$

cuius ſeriei poſito pro n valore debito, ſumma mox reperietur $= 0{,}0086861791849$ quae addita ad $l99 = 1{,}9956351945979$ oritur
$l101 = 2{,}0043213737829$

81.

81. Tribuamus nunc in expreſſione noſtra generali y valorem exponentialem, ſitque $y = a^x$, poſito $x + \omega$ loco x; erit $z = a^{x+\omega}$, cuius valor ob differentialia:

$$\frac{dy}{dx} = a^x\, la\,;\ \frac{ddy}{dx^2} = a^x (la)^2\,;\ \frac{d^3y}{dx^3} = a^x (la)^3\,;\ \text{\&c. erit}$$

$$a^{x+\omega} = a^x\left(1 + \frac{\omega la}{1} + \frac{\omega^2 (la)^2}{1.\,2} + \frac{\omega^3 (la)^3}{1.\,2.\,3} + \text{\&c.}\right)$$

quae ſi dividatur per a^x prodibit ſeries valores quantitatis exponentialis exprimens, quam ſupra in Introductione iam elicuimus: nempe

$$a^\omega = 1 + \frac{\omega la}{1} + \frac{\omega^2 (la)^2}{1.\,2} + \frac{\omega^3 (la)^3}{1.\,2.\,3} + \frac{\omega^4 (la)^4}{1.\,2.\,3.\,4} + \text{\&c.}$$

Simili modo ſumto ω negativo erit:

$$a^{-\omega} = 1 - \frac{\omega la}{1} + \frac{\omega^2 (la)^2}{1.\,2} - \frac{\omega^3 (la)^3}{1.\,2.\,3.} + \frac{\omega^4 (la)^4}{1.\,2.\,3.\,4} - \text{\&c.}$$

ex quarum combinatione oritur

$$\frac{a^\omega + a^{-\omega}}{2} = 1 + \frac{\omega^2 (la)^2}{1.\,2} + \frac{\omega^4 (la)^4}{1.\,2.\,3.\,4} + \frac{\omega^6 (la)^6}{1.\,2 \ldots 6} + \text{\&c.}$$

$$\frac{a^\omega - a^{-\omega}}{2} = \frac{\omega la}{1} + \frac{\omega^3 (la)^3}{1.\,2.\,3} + \frac{\omega^5 (la)^5}{1.2.3.4.5} + \text{\&c.}$$

Ubi notandum eſt la denotare logarithmum hyperbolicum numeri a.

82. Huius formulae ope ex dato quovis logarithmo numerus ei conveniens reperiri poterit. Sit enim propoſitus logarithmus quicunque u ad canonem, in quo numeri a logarithmus $= 1$ ſtatuitur, pertinens. Quaeratur in eodem canone logarithmus x proxime ad u accedens, ſitque $u = x + \omega$; numerus autem logarithmo x conveniens ſit $= y = a^x$, erit numerus logarithmo $u = x + \omega$ reſpondens $= a^{x+\omega} = z$; fietque

$$z = y\left(1 + \frac{\omega la}{1} + \frac{\omega^2 (la)^2}{1.\,2} + \frac{\omega^3 (la)^3}{1.\,2.\,3} + \frac{\omega^4 (la)^4}{1.\,2.\,3.\,4} + \text{\&c.}\right)$$

quae ſeries ob ω numerum valde parvum, vehementer converget, cuius uſum ſequenti exemplo declaremus.

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EXEMPLUM.

Quaeratur numerus iſti binarii poteſtati, $2^{2^{24}}$ *aequalis*.

Cum ſit $2^{24} = 16777216$, erit $2^{2^{24}} = 2^{16777216}$, ſumendiſque logarithmis vulgaribus, erit huius numeri logarithmus $= 16777216\, l2$. Cum autem ſit:

$$l2 = 0,30102999566398119521373889$$

numeri quaeſiti logarithmus erit:

$$5050445,25973367593203906 3$$

cuius characteriſtica indicat numerum quaeſitum exprimi 5050446 figuris, quae cum omnes exhiberi nequeant, ſufficiet figuras initiales aſſignaſſe, quae ex mantiſſa

$$,2597336759320390 63 = u$$

inveſtigari debent. Ex tabulis autem colligitur, numerum cuius logarithmus proxime ad hunc accedat ſore $18.101 = 1,818$; qui ponatur y; cuius logarithmus

$x = 0,25959387888594864 4$, unde erit

$\omega = 0,00013979704609041 9$. Cum iam ſit

$a = 10$ erit

$la = 2,30258509299404568401799 14$ &

$\omega la = 0,000321894594372398$ Deinde erit

$$y = 1,818000000000000000$$

$$\frac{\omega la}{1} y = \quad 585204372569020$$

$$\frac{\omega^2 (la)^2}{1 \cdot 2} y = \quad 94187062064$$

$$\frac{\omega^3 (la)^3}{1 \cdot 2 \cdot 3} y = \quad 10106100$$

$$\frac{\omega^4 (la)^4}{1 \cdot 2 \cdot 3 \cdot 4} y = \quad 813$$

$$18185852985697 37997$$

haeque ſunt figurae initiales numeri quaeſiti, cuius omnes figurae excepta ſorte ultima ſunt iuſtae.

83. Conſideremus quantitates tranſcendentes a circulo pendentes, ſitque uti perpetuo ponimus, radius circuli $= 1$, atque y denotet arcum circuli cuius ſinus $= x$ ſeu ſit $y = \text{A ſin}\, x$. Ponatur $x + \omega$ loco x, eritque $z = \text{A ſin}\,(x + \omega)$: ad quem valorem exprimendum quaerantur differentialia ipſius y:

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1-xx)}} \quad ; \quad \frac{ddy}{dx^2} = \frac{+x}{(1-xx)^{\frac{3}{2}}} \quad ; \quad \frac{d^3y}{dx^3} = \frac{1+2xx}{(1-xx)^{\frac{5}{2}}}$$

$$\frac{d^4y}{dx^4} = \frac{9x+6x^3}{(1-xx)^{\frac{7}{2}}} \quad ; \quad \frac{d^5y}{dx^5} = \frac{9+72x^2+24x^4}{(1-xx)^{\frac{9}{2}}}$$

$$\frac{d^6y}{dx^6} = \frac{225x+600x^3+120x^5}{(1-xx)^{\frac{11}{2}}} \quad ; \quad \&c.$$

Ex his ergo invenitur:

$$\text{A ſin}\,(x+\omega) = \text{A ſin}\, x + \frac{\omega}{\sqrt{(1-xx)}} + \frac{\omega^2 x}{2(1-xx)^{\frac{3}{2}}} + \frac{\omega^3(1+2xx)}{6(1-xx)^{\frac{5}{2}}}$$
$$+ \frac{\omega^4(9x+6x^3)}{24(1-xx)^{\frac{7}{2}}} + \frac{\omega^5(9+72x^2+24x^4)}{120(1-xx)^{\frac{9}{2}}} + \&c.$$

84. Si ergo cognitus fuerit arcus, cuius ſinus eſt $= x$, huius formulae beneficio inveniri poterit arcus, cuius ſinus eſt $x + \omega$, ſi fuerit ω quantitas valde parva. Series autem cuius ſumma addi debet, exprimetur in partibus radii, quae ad arcum facile reducentur: uti ex hoc exemplo intelligetur.

EXEMPLUM.

Quaeratur arcus circuli, cuius ſinus eſt $= \frac{1}{3} = 0,3333333333$.

Quaeratur ex tabulis ſinuum arcus, cuius ſinus ſit proxime minor, quam $\frac{1}{3}$, qui erit $19°, 28'$, cuius ſinus eſt $= 0,3332584$. Statuatur ergo $19°, 28', = \text{A ſin}\, x = y$ erit $x = 0,3332584$, & $\omega = 0,0000749$, atque ex tabulis $\sqrt{(1-xx)} = \text{coſ}\, y = 0,9428356$. Erit ergo arcus quaeſitus z, cuius ſinus $= \frac{1}{3}$ proponitur $= 19°, 28' + \frac{\omega}{\text{coſ}\, y} + \frac{\omega\omega\, \text{ſin}\, y}{2\, \text{coſ}\, y^3}$, quae expreſſio iam ſufficit; erit ergo per logarithmos calculum inſtituendo:

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$$l\omega = 5,8744818$$
$$l\operatorname{coſ} y = 9,9744359$$
$$l\frac{\omega}{\operatorname{coſ} y} = 5,9000459 \quad ; \quad \frac{\omega}{\operatorname{coſ} y} = 0,0000794412$$
$$l\frac{\omega^2}{\operatorname{coſ} y^2} = 1,8000918$$
$$l\frac{\operatorname{ſin} y}{\operatorname{coſ} y} = 9,5483452$$
$$1,3484370$$
$$l2 = 0,3010300$$
$$l\frac{\omega^2 \operatorname{ſin} y}{2\operatorname{coſ} y^3} = 1,0474070 \quad ; \quad \frac{\omega^2 \operatorname{ſin} y}{2\operatorname{coſ} y^3} = 0,0000000011$$
$$\text{Summa} = 0,0000794423$$

qui eſt valor arcus ad 19°, $28'$ addendi, ad quem in minutis ſecundis exprimendum, ſumamus eius logarithmum

qui eſt	5,9000518
a quo ſubtrahatur	4,6855749
	1,2144769
cui log. reſpondet num. =	16,38615

qui eſt numerus minutorum ſecundorum; fractionem vero in tertiis & quartis exprimendo fiet arcus quaeſitus

$$= 19^\circ,\ 28',\ 16'',\ 23''',\ 10^{\text{iv}},\ 8^{\text{v}},\ 24^{\text{vi}}.$$

85. Simili modo expreſſio pro coſinibus eruetur; poſito enim $y = \text{A} \operatorname{coſ} x$; quia eſt $dy = \frac{-dx}{\sqrt{(1-xx)}}$, ſeries ante inventa invariata manebit, dummodo eius ſigna permutentur. Erit itaque

$$\text{A} \operatorname{coſ}(x+\omega) = \text{A} \operatorname{coſ} x - \frac{\omega}{\sqrt{(1-xx)}} - \frac{\omega^2 x}{2(1-xx)^{\frac{3}{2}}} - \frac{\omega^3(1+2xx)}{6(1-xx)^{\frac{5}{2}}}$$
$$- \frac{\omega^4(9x+6x^3)}{24(1-xx)^{\frac{7}{2}}} - \frac{\omega^5(9+72x^2+24x^4)}{120(1-xx)^{\frac{9}{2}}} - \&c.$$

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quae ſeries pariter ac praecedens vehementer ſemper converget, ſi ex tabulis ſinuum proxime veri anguli excerpantur, ita ut plerumque unicus terminus primus $\frac{\omega}{\sqrt{(1-xx)}}$ ſufficiat. Interim tamen ſi x fuerit ipſi 1 ſeu ſinui toti proxime aequalis, tum ob denominatores admodum parvos illa ſeries convergentiam amittit. His igitur caſibus, quibus x non multum ab 1 deficit, quoniam tum differentiae fiunt minimae, commodius utemur ſolita interpolatione.

86. Ponamus quoque pro y arcum cuius tangens datur, ſitque $y = \text{A tang}\, x$ & $z = \text{A tang}\,(x+\omega)$ ita ut ſit

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3 y}{6dx^3} + \&c.$$

Ad quos terminos indagandos quaerantur ipſius y ſingula differentialia:

$$\frac{dy}{dx} = \frac{1}{1+xx}\;;\quad \frac{ddy}{dx^2} = \frac{-2x}{(1+xx)^2}\;;\quad \frac{d^3y}{dx^3} = \frac{-2+6xx}{(1+xx)^3}$$

$$\frac{d^4y}{dx^4} = \frac{24x-24x^3}{(1+xx)^4}\;;\quad \frac{d^5y}{dx^5} = \frac{24-240x^2+120x^4}{(1+xx)^5}\;;$$

$$\frac{d^6y}{dx^6} = \frac{-720x+2400x^3-720x^5}{(1+xx)^6}\;;\quad \&c.$$

unde colligitur fore: $\text{A tang}\,(x+\omega) = \text{A tang}\, x +$

$$\frac{\omega}{1(1+xx)} - \frac{\omega^2 x}{(1+xx)^2} + \frac{\omega^3}{(1+xx)^3}\left(xx-\tfrac{1}{3}\right) - \frac{\omega^4}{(1+xx)^4}\left(x^3-x\right) +$$

$$\frac{\omega^5}{(1+xx)^5}\left(x^4-2x^2+\tfrac{1}{5}\right) - \frac{\omega^6}{(1+xx)^6}\left(x^5-\tfrac{10}{3}x^3+x\right) + \&c.$$

87. Haec ſeries, cuius lex progreſſionis non adeo manifeſta eſt, tranſmutari poteſt in aliam formam, cuius progreſſio ſtatim in oculos incurrit. Ponatur in hunc finem $\text{A tang}\, x = 90^\circ - u$, ut ſit $x = \cot u = \frac{\text{coſ}\, u}{\text{ſin}\, u}$; erit

1 +

$1+xx=\frac{1}{\sin u^2}$, unde ſit $\frac{dy}{dx}=\frac{1}{1+xx}=\sin u^2$. Cum deinde ſit $dx=\frac{-du}{\sin u^2}$, ſeu $du=-dx\sin u^2$, fiet ulteriora differentialia ſumendo:

$$\frac{ddy}{dx}=2du\sin u\cos u=du\sin 2u=-dx\sin u^2.\sin 2u$$

ideoque $\frac{ddy}{1dx^2}=-\sin u^2.\sin 2u.$

$$\frac{d^3y}{2dx^2}=-du\sin u.\cos u.\sin 2u-du\sin u^2\cos 2u=-du\sin u.\sin 3u$$
$$=dx\sin u^3\sin 3u$$

ideoque $\frac{d^3y}{1.2dx^3}=+\sin u^3.\sin 3u$

$$\frac{d^4y}{1.2.3dx^3}=du\sin u^3.(\cos u.\sin 3u+\sin u.\cos 3u)=du\sin u^3.\sin 4u$$
$$=-dx\sin u^4.\sin 4u$$

ideoque $\frac{d^4y}{1.2.3dx^4}=-\sin u^4.\sin 4u$

$$\frac{d^5y}{1.2.3.4dx^4}=-du\sin u^4(\cos u\sin 4u+\sin u.\cos 4u)=-du.\sin u^4.\sin 5u$$
$$=+dx\sin u^5.\sin 5u$$

ideoque $\frac{d^5y}{1.2.3.4dx^5}=+\sin u^5.\sin 5u$

&c.

Ex quibus colligitur fore:

$$A\,tg(x+\omega)=A\,tg\,x+\frac{\omega}{1}\sin u.\sin u-\frac{\omega^2}{2}\sin u^2.\sin 2u+\frac{\omega^3}{3}\sin u^3.\sin 3u$$
$$-\frac{\omega^4}{4}\sin u^4\sin 4u+\frac{\omega^5}{5}\sin u^5.\sin 5u-\frac{\omega^6}{6}\sin u^6\sin 6u+\&c.$$

ubi

ubi cum ſit A tg $x = y$ & A tg $x = 90° - u$, erit $y = 90° - u$.

88. Si ponatur A cot $x = y$ & A cot $(x+\omega) = z$; erit

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{1.2dx^2} + \frac{\omega^3 d^3 y}{1.2.3dx^3} + \frac{\omega^4 d^4 y}{1.2.3.4dx^4} + \&c.$$

Cum autem ſit $dy = \frac{-dx}{1+xx}$, termini huius ſeriei congruent praeter primum cum ante inventis, exceptis tantum ſignis. Quare ſi ponatur, ut ante A tang $x = 90° - u$, ſeu A cot $x = u$, ut ſit $u = y$; erit:

$$A\cot(x+\omega) = A\cot x - \frac{\omega}{1}\sin u.\sin u + \frac{\omega^2}{2}\sin u^2.\sin 2u - \frac{\omega^3}{3}\sin u^3 \sin 3u$$

$$+ \frac{\omega^4}{4}\sin u^4.\sin 4u - \frac{\omega^5}{5}\sin u^5.\sin 5u + \&c.$$

quae expreſſio immediate ex praecedente ſequitur: quia enim eſt A cot $(x+\omega) = 90 -$ A tang $(x+\omega)$

& A cot $x = 90 -$ A tang$x$; erit

A cot $(x+\omega) -$ A cot $x = -$ A tang $(x+\omega) +$ A tang x.

89. Ex his expreſſionibus multa egregia corollaria conſequuntur, prout loco x & ω dati valores ſubſtituantur. Sit igitur primum $x = 0$; & cum ſit $u = 90° -$ A tang x fiet $u = 90°$; atque $\sin u = 1$; $\sin 2u = 0$; $\sin 3u = -1$; $\sin 4u = 0$; $\sin 5u = 1$; $\sin 6u = 0$; $\sin 7u = -1$; &c. unde fiet

$$A\tang \omega = \frac{\omega}{1} - \frac{\omega^3}{3} + \frac{\omega^5}{5} - \frac{\omega^7}{7} + \frac{\omega^9}{9} - \frac{\omega^{11}}{11} + \&c.$$

quae eſt notiſſima ſeries exprimens arcum, cuius tangens eſt $= \omega$. Sit $x = 1$, erit A tang $x = 45°$, ideoque $u = 45°$, hinc

$\sin u = \frac{1}{\sqrt{2}}$; $\sin 2u = 1$; $\sin 3u = \frac{1}{\sqrt{2}}$; $\sin 4u = 0$;

$\sin 5u = -\frac{1}{\sqrt{2}}$; $\sin 6u = -1$; $\sin 7u = -\frac{1}{\sqrt{2}}$; $\sin 8u = 0$

$\sin 9u = \frac{1}{\sqrt{2}}$ &c. Ex quibus fit:

$$A\,tg(1+\omega) = 45^\circ + \frac{\omega}{2} - \frac{\omega^2}{2.2} + \frac{\omega^3}{3.4} - \frac{\omega^5}{5.8} + \frac{\omega^6}{6.8} - \frac{\omega^7}{7.16} + \frac{\omega^9}{9.32} - \frac{\omega^{10}}{10.32} + \frac{\omega^{11}}{11.64} - \frac{\omega^{13}}{13.128} + \frac{\omega^{14}}{14.128} - \&c.$$

Si igitur ſit $\omega = -1$; ob $A tang(1+\omega) = 0$, & $45^\circ = \frac{\pi}{4}$ fiet:

$$\frac{\pi}{4} = \frac{1}{1.2} + \frac{1}{2.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} - \frac{1}{6.2^3} - \frac{1}{7.2^4} + \frac{1}{9.2^5} + \frac{1}{10.2^5} + \frac{1}{11.2^6} - \&c.$$

qui valor ſi loco arcus 45° ſubſtituatur in illa expreſſione erit: $A tang(1+\omega) =$

$$\frac{\omega+1}{1.2} - \frac{\omega^2+1}{2.2} + \frac{\omega^3+1}{3.2^2} - \frac{\omega^5-1}{5.2^3} + \frac{\omega^6-1}{6.2^3} - \frac{\omega^7-1}{7.2^4} + \&c.$$

Illa autem ſeries maxime eſt idonea ad valorem ipſius $\frac{\pi}{4}$ proxime inveniendum Cum ſit

$$\frac{\pi}{4} = \frac{1}{1.2} + \frac{1}{2.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} - \frac{1}{6.2^3} - \frac{1}{7.2^4} + \&c.$$

termini autem in denominatoribus habentes 2, 6, 10, &c.

$\frac{1}{2.2} - \frac{1}{6.2^3} + \frac{1}{10.2^5} - \frac{1}{14.2^7} + \&c.$ exprimunt $\frac{1}{2} A\,tg\,\frac{1}{2}$; erit

$$\frac{\pi}{4} = \tfrac{1}{2} A\,tang\,\tfrac{1}{2} + \frac{1}{1.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} - \frac{1}{7.2^4} + \frac{1}{9.2^5} + \frac{1}{11.2^6}\ \&c.$$

In altera autem formula poſito ω negativo, cum ſit

$$A\,tang(1-\omega) = \begin{cases} + \frac{1}{1.2} + \frac{1}{2.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} - \frac{1}{6.2^3} - \frac{1}{7.2^4} + \&c. \\ - \frac{\omega}{1.2} - \frac{\omega^2}{2.2} - \frac{\omega^3}{3.2^2} + \frac{\omega^5}{5.2^3} + \frac{\omega^6}{6.2^3} + \frac{\omega^7}{7.2^4} - \&c. \end{cases}$$

ſi fiat $\omega = \frac{1}{2}$; erit:

A

$$\text{A tang}\,\tfrac{1}{2} = \begin{cases} \frac{1}{1.2} + \frac{1}{2.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} - \frac{1}{6.2^3} - \frac{1}{7.2^4} + \&c. \\ -\frac{1}{1.2^2} - \frac{1}{2.2^3} - \frac{1}{3.2^5} + \frac{1}{5.2^8} + \frac{1}{6.2^9} + \frac{1}{7.2^{11}} - \&c. \end{cases}$$

& terminis per 2, 6, 10, &c. diviſis ſeorſim ſumtis erit

$$\text{A tg}\,\tfrac{1}{2} = \tfrac{1}{2}\text{A tg}\,\tfrac{1}{2} + \frac{1}{1.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} - \frac{1}{7.2^4} + \frac{1}{9.2^5} + \&c.$$

$$- \tfrac{1}{2}\text{A tang}\,\tfrac{1}{4} - \frac{1}{1.2^2} - \frac{1}{3.2^5} + \frac{1}{5.2^8} + \frac{1}{7.2^{11}} - \frac{1}{9.2^{14}} - \&c.$$

ideoque $\quad \tfrac{1}{2}\text{A tang}\,\tfrac{1}{2} = \frac{1}{1.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} - \frac{1}{7.2^4} + \&c.$

$$- \tfrac{1}{2}\text{A tang}\,\tfrac{1}{4} - \frac{1}{1.2^2} - \frac{1}{3.2^5} + \frac{1}{5.2^8} + \frac{1}{7.2^{11}} - \&c.$$

qui valor ſi in ſuperiore ſerie ſubſtituatur, atque A tang $\frac{1}{4}$ ipſe in ſeriem convertatur, reperietur

$$\frac{\pi}{4} = \begin{cases} 1 + \frac{1}{3.2^1} - \frac{1}{5.2^2} - \frac{1}{7.2^3} + \frac{1}{9.2^4} + \&c. \\ -\frac{1}{1.2^2} - \frac{1}{3.2^5} + \frac{1}{5.2^8} + \frac{1}{7.2^{11}} - \frac{1}{9.2^{14}} - \&c. \\ -\frac{1}{1.2^4} + \frac{1}{3.2^{10}} - \frac{1}{5.2^{16}} + \frac{1}{7.2^{22}} - \frac{1}{9.2^{28}} + \&c. \end{cases}$$

90. Sequuntur hae multaeque aliae ſeries ex poſitione $x = 1$: ſin autem ponamus $x = \sqrt{3}$, ut ſit A tang $x = 60°$, fiet $u = 30$, & ſin $u = \frac{1}{2}$; ſin $2u = \frac{\sqrt{3}}{2}$; ſin $3u = 1$; ſin $4u = \frac{\sqrt{3}}{2}$; ſin $5u = \frac{1}{2}$; ſin $6u = 0$; ſin $7u = -\frac{1}{2}$; &c. unde erit:

$$\text{A tang}(\sqrt{3} + \omega) = 60° + \frac{\omega}{1.2^2} - \frac{\omega^2\sqrt{3}}{2.2^3} + \frac{\omega^3}{3.2^3} - \frac{\omega^4\sqrt{3}}{4.2^5} +$$

$$+\frac{\omega^5}{5.2^6}-\frac{\omega^7}{7.2^8}+\frac{\omega^8\sqrt{3}}{8.2^9}-\frac{\omega^9}{9.2^9}+\frac{\omega^{10}\sqrt{3}}{10.2^{11}}-\frac{\omega^{11}}{11.2^{12}}+\ \&c.$$

Sin autem ponatur $x=\frac{1}{\sqrt{3}}$, ut ſit $A\ \mathrm{tang}\, x = 30°$; erit $u = 60°$; atque $\sin u = \frac{\sqrt{3}}{2}$; $\sin 2u = \frac{\sqrt{3}}{2}$; $\sin 3u = 0$; $\sin 4u = -\frac{\sqrt{3}}{2}$; $\sin 5u = -\frac{\sqrt{3}}{2}$; $\sin 6u = 0$; $\sin 7u = \frac{\sqrt{3}}{2}$; &c.

quibus valoribus ſubſtitutis erit:

$$A\ \mathrm{tg}\left(\frac{1}{\sqrt{3}}+\omega\right)=30°+\frac{3\omega}{1.2^2}-\frac{3\omega^2\sqrt{3}}{2.2^3}+\frac{3^2\omega^4\sqrt{3}}{4.2^5}-\frac{3^3\omega^5}{5.2^6}+\ \&c.$$

ſi igitur ſit $\omega=-\frac{1}{\sqrt{3}}$, ob $30° = \frac{\pi}{6}$; erit:

$$\frac{\pi}{6\sqrt{3}}=\frac{1}{1.2^2}+\frac{1}{2.2^3}-\frac{1}{4.2^5}-\frac{1}{5.2^6}+\frac{1}{7.2^8}+\frac{1}{8.2^9}-\ \&c.$$

91. Reſumamus expreſſionem generalem inventam:

$$A\ \mathrm{tang.}\,(x+\omega) = A\ \mathrm{tang}\, x$$
$$+\frac{\omega}{1}\sin u.\sin u-\frac{\omega^2}{2}\sin u^2.\sin 2u+\frac{\omega^3}{3}\sin u^3.\sin 3u-\ \&c.$$

ac ponamus $\omega = -x$, ut ſit $A\ \mathrm{tang}(x+\omega)=0$, eritque

$$A\ \mathrm{tang}\, x =$$
$$\frac{x}{1}\sin u.\sin u+\frac{x^2}{2}\sin u^2.\sin 2u+\frac{x^3}{3}\sin u^3.\sin 3u+\ \&c.$$

Cum autem ſit $A\ \mathrm{tang}\, x = 90 - u = \frac{\pi}{2} - u$;

erit: $x = \cot u = \frac{\cos u}{\sin u}$. Quamobrem erit:

$$\frac{\pi}{2}=u+\cos u.\sin u+\tfrac{1}{2}\cos u^2.\sin 2u+\tfrac{1}{3}\cos u^3.\sin 3u+\tfrac{1}{4}\cos u^4.\sin 4u+\ \&c.$$

quae

quae ſeries eo magis eſt notatu digna, quod quicunque arcus loco u accipiatur, valor ſeriei ſemper prodeat idem $= \frac{\pi}{2}$. Sin autem ſit $\omega = -2x$, ob $A\,\text{tang}(-x) = -A\,\text{tg}\,x$;

fiet: $2\,A\,\text{tang}\,x =$

$$\frac{2x}{1}\,\text{ſin}\,u.\,\text{ſin}\,u + \frac{4x^2}{2}\,\text{ſin}\,u^2.\,\text{ſin}\,2u + \frac{8x^3}{3}\,\text{ſin}\,u^3.\,\text{ſin}\,3u + \&c.$$

Cum autem ſit $A\,\text{tang}\,x = \frac{\pi}{2} - u$ & $x = \frac{\text{coſ}u}{\text{ſin}u}$, erit:

$$\pi = 2u + \frac{2}{1}\,\text{coſ}u.\,\text{ſin}u + \frac{2^2}{2}\,\text{coſ}u^2.\text{ſin}\,2u + \frac{2^3}{3}\,\text{coſ}u^3.\,\text{ſin}\,3u + \&c.$$

Sit $u = 45° = \frac{\pi}{4}$; erit $\text{coſ}u = \frac{1}{\sqrt{2}}$; $\text{ſin}u = \frac{1}{\sqrt{2}}$; $\text{ſin}2u = 1$;

$\text{ſin}\,3u = \frac{1}{\sqrt{2}}$; $\text{ſin}\,4u = 0$; $\text{ſin}\,5u = \frac{-1}{\sqrt{2}}$; $\text{ſin}\,6u = -1$;

$\text{ſin}\,7u = \frac{-1}{\sqrt{2}}$; $\text{ſin}\,8u = 0$; $\text{ſin}\,9u = \frac{1}{\sqrt{2}}$; eritque

$$\frac{\pi}{2} = \frac{1}{1} + \frac{2}{2} + \frac{2}{3} - \frac{2^2}{5} - \frac{2^3}{6} - \frac{2^3}{7} + \frac{2^4}{9} + \frac{2^5}{10} + \frac{2^5}{11} - \&c.$$

quae ſeries etſi divergit, tamen ob ſimplicitatem eſt notatu digna.

92. Ponatur in expreſſione generali inventa:

$\omega = -x - \frac{1}{x} = \frac{-1}{\text{ſin}\,u.\,\text{coſ}\,u}$, ob $x = \frac{\text{coſ}\,u}{\text{ſin}\,u}$; erit:

$$A\,\text{tang}(x + \omega) = A\,\text{tang} - \frac{1}{x} = -A\,\text{tang}\,\frac{1}{x} = -\frac{\pi}{2} + A\,\text{tang}\,x.$$

Hinc ergo obtinebitur ſequens expreſſio:

$$\frac{\pi}{2} = \frac{\text{ſin}\,u}{1\,\text{coſ}u} + \frac{\text{ſin}2\,u}{2\,\text{coſ}u^2} + \frac{\text{ſin}3\,u}{3\text{coſ}u^3} + \frac{\text{ſin}4\,u}{4\text{coſ}u^4} + \frac{\text{ſin}5\,u}{5\text{coſ}u^5} + \&c.$$

quae poſito $u = 45°$ dat eandem ſeriem, quam ultimo loco invenimus. Sin autem ponamus $\omega = -\sqrt{(1 + xx)}$

ob

ob $x = \frac{\cos u}{\sin u}$, fiet $\omega = -\frac{1}{\sin u}$, &

$$A \tang [x - \sqrt{(1 + xx)}] - A \tang [\sqrt{(1 + xx)} - x]$$
$$= -\tfrac{1}{2} A \tang \frac{1}{x} = -\tfrac{1}{2}\left(\frac{\pi}{2} - A \tang x\right) = -\tfrac{1}{2} u,$$

& $A \tang x = \frac{\pi}{2} - u$. Hancobrem erit:

$$\frac{\pi}{2} = \tfrac{1}{2} u + \tfrac{1}{1} \sin u + \tfrac{1}{2} \sin 2u + \tfrac{1}{3} \sin 3u + \tfrac{1}{4} \sin 4u + \&c.$$

Quodſi haec aequatio differentietur erit:

$$0 = \tfrac{1}{2} + \cos u + \cos 2u + \cos 3u + \cos 4u + \cos 5u + \&c.$$

cuius ratio ex natura ſerierum recurrentium intelligitur.

93. Si ſimili modo ſeries ante inventae differentientur, novae ſeries ſummabiles reperientur. Ac primo quidem ex ſerie:

$$A \tang (1+\omega) = \frac{\pi}{4} + \frac{\omega}{2} - \frac{\omega^2}{2.2} + \frac{\omega^3}{3.4} - \frac{\omega^5}{5.8} + \frac{\omega^6}{6.8} - \&c.$$

ſequitur

$$\frac{1}{2+2\omega+\omega^2} = \tfrac{1}{2} - \frac{\omega}{2} + \frac{\omega^2}{4} - \frac{\omega^4}{8} + \frac{\omega^5}{8} - \frac{\omega^6}{16} + \frac{\omega^8}{32} - \&c.$$

quae oritur ex evolutione fractionis $\frac{2-2\omega+\omega^2}{4+\omega^4} = \frac{1}{2+2\omega+\omega^2}$;

Deinde iſta ſeries:

$$\frac{\pi}{2} = u + \cos u \sin u + \tfrac{1}{2} \cos u^2 . \sin 2u + \tfrac{1}{3} \cos u^3 \sin 3u + \tfrac{1}{4} \cos u^4 \sin 4u + \&c.$$

per differentiationem dabit:

$$0 = 1 + \cos 2u + \cos u . \cos 3u + \cos u^2 . \cos 4u + \cos u^3 \cos 5u + \&c.$$

Denique ſeries $\frac{\pi}{2} = \frac{\sin u}{\cos u} + \frac{\sin 2u}{2\cos u^2} + \frac{\sin 3u}{3\cos u^3} + \frac{\sin 4u}{4\cos u^4} + \&c.$

dat $0 = \frac{1}{\cos u^2} + \frac{\cos u}{\cos u^3} + \frac{\cos 2u}{\cos u^4} + \frac{\cos 3u}{\cos u^5} + \frac{\cos 4u}{\cos u^6} + \&c.$

ſeu

ſeu $0 = 1 + \frac{\text{coſ}\, u}{\text{coſ}\, u} + \frac{\text{coſ}\, 2u}{\text{coſ}\, u^2} + \frac{\text{coſ}\, 3u}{\text{coſ}\, u^3} + \frac{\text{coſ}\, 4u}{\text{coſ}\, u^4} + \&c.$

94. Imprimis autem expreſſio inventa:

$$A\,\text{tang}\,(x+\omega) =$$

$$A\,\text{tang}\,x + \frac{\omega}{1}\,\text{ſin}\,u.\,\text{ſin}\,u - \frac{\omega^2}{2}\,\text{ſin}\,u^2.\,\text{ſin}\,2u + \frac{\omega^3}{3}\,\text{ſin}\,u^3.\,\text{ſin}\,3u - \&c.$$

exiſtente $x = \text{cot}\,u$ ſeu $u = A\,\text{cot}\,x = 90° - A\,\text{tang}\,x$ inſerviet ad angulum ſeu arcum datae cuique tangenti reſpondentem inveniendum. Sit enim propoſita tangens $= t$, quaeraturque in tabulis tangens ad hanc proxime accedens $= x$, cui reſpondeat arcus $= y$; eritque $u = 90° - y$. Tum ponatur $x + \omega = t$, ſeu $\omega = t - x$; eritque arcus quaeſitus:

$$= y + \frac{\omega}{1}\,\text{ſin}\,u.\,\text{ſin}\,u - \frac{\omega^2}{2}\,\text{ſin}\,u^2.\,\text{ſin}\,2u + \&c.$$

quae regula tum praecipue eſt utilis, cum tangens propoſita fuerit admodum magna, ac propterea arcus quae ſitus parum a 90° diſcrepet. His enim caſibus ob tangentes vehementer increſcentes, ſolita methodus interpolationum nimium a veritate abducit. Sit ergo propoſitum hoc exemplum.

EXEMPLUM.

Quaeratur arcus, cuius tangens ſit = 100, *poſito radio* = 1. Arcus proxime quaeſito aequalis eſt 89°, 25′, cuius tangens

eſt	$x =$ 98, 217943	ſecund.
quae ſubtrahatur a	$t =$ 100, 000000	
remanebit	$\omega =$ 1, 782057	

Deinde cum ſit $y = 89°, 25'$, erit $u = 0°, 35'$, $2u = 1°, 10'$, $3u = 1°; 45'$, &c. Iam ſinguli termini per logarithmos inveſtigentur.

Ad

Ad	$l\omega =$	0,2509215	
add.	$l\,\text{ſin}\,u =$	8,0077867	
	$l\,\text{ſin}\,u =$	8,0077867	
	$l\,\omega\,\text{ſin}\,u.\,\text{ſin}\,u =$	6,2664949	
		4,6855749	
ſubtr.	$=$	1,5809200	
Ergo	$\omega\,\text{ſin}\,u.\,\text{ſin}\,u =$	38,09956	ſecund.
Ad	$l\,\omega\,\text{ſin}\,u^2 =$	6,2664949	
add.	$l\omega =$	0,2509215	
	$l\,\text{ſin}\,2u =$	8,3087941	
		4,8262105	
ſubtr.	$l\,2 =$	0,3010300	
	$l\,\frac{1}{2}\,\omega^2\,\text{ſin}\,u^2.\,\text{ſin}\,2u =$	4,5251805	
ſubtr.		4,6855749	
Remanet		9,8396056	
Ergo	$\frac{1}{2}\,\omega^2\,\text{ſin}\,u^2.\,\text{ſin}\,2u =$	0,69120	ſecund.
Porro ad	$l\omega^3 =$	0,7527645	
add.	$l\,\text{ſin}\,u^3 =$	4,0233601	
	$l\,\text{ſin}\,3u =$	8,4848479	
		3,2609725	
ſubtr.	$l\,3 =$	0,4771213	
		2,7838512	
ſubtr.		4,6855749	
		8,0982763	
Ergo	$\frac{1}{3}\,\omega^3\,\text{ſin}\,u^3\,\text{ſin}\,3u =$	0,01254	ſecund.
Denique ad	$l\omega^4 =$	1,0036860	
add.	$l\,\text{ſin}\,u^4 =$	2,0311468	
	$l\,\text{ſin}\,4u =$	8,6097341	
		1,6445669	
ſubtr.	$l\,4 =$	0,6020600	
		1,0425069	
ſubtr.		4,6855749	
		6,3569320	
Ergo	$\frac{1}{4}\,\omega^4\,\text{ſin}\,u^4\,\text{ſin}\,4u =$	0,00023	ſecund.

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Hinc:

Termini addendi	Termini ſubtrahendi
38,09956	0,69120
0,01254	0,00023
38,11210	0,69143
ſubtr. 0,69143	

$37,42067 = 37^{II}, 25^{III}, 14^{IV}, 24^{V}, 36^{VI}.$

Quocirca arcus, cuius tangens centies ſuperat radium erit:

$89^{\circ}, 25^{I}, 37^{II}, 25^{III}, 14^{IV}, 24^{V}, 36^{VI},$

neque error ad minuta quarta aſcendit; ſed in minutis tantum quintis ineſſe poteſt, ex quo vere hunc angulum pronunciare poterimus $= 89^{\circ}, 25^{I}, 37^{II}, 25^{III}, 14^{IV}$. Si tangens adhuc maior proponatur, etiamſi fortaſſe ω maius prodeat, tamen ob u angulum adhuc minorem, aeque expedite arcus definiri poterit.

95. Cum hic pro y arcum circuli ſubſtituerimus, nunc functiones reciprocas in locum y ponamus, cuiuſmodi ſunt $\sin x$, $\cos x$, $\tang x$, $\cot x$, &c. Sit igitur $y = \sin x$, poſitoque $x + \omega$ loco x, fiet: $z = \sin(x + \omega)$, atque aequatio

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \&c.$$

ob $\frac{dy}{dx} = \cos x$; $\frac{ddy}{dx^2} = -\sin x$; $\frac{d^3y}{dx^3} = -\cos x$; &c. dabit

$$\sin(x + \omega) = \sin x + \omega \cos x - \tfrac{1}{2}\omega^2 \sin x - \tfrac{1}{6}\omega^3 \cos x + \tfrac{1}{24}\omega^4 \sin x + \&c.$$

& ſumto ω negativo erit:

$$\sin(x - \omega) = \sin x - \omega\cos x - \tfrac{1}{2}\omega^2 \sin x + \tfrac{1}{6}\omega^3 \cos x + \tfrac{1}{24}\omega^4 \sin x - \&c.$$

Quod ſi vero ſtatuatur $y = \cos x$,

ob $\frac{dy}{dx} = -\sin x$; $\frac{ddy}{dx^2} = -\cos x$; $\frac{d^3y}{dx^3} = \sin x$; $\frac{d^4y}{dx^4} = \cos x$; &c.

erit:

$$\cos(x + \omega) = \cos x - \omega \sin x - \tfrac{1}{2}\omega^2 \cos x + \tfrac{1}{6}\omega^3 \sin x + \tfrac{1}{24}\omega^4 \cos x - \&c.$$

& facto ω negativo erit:

$$\cos(x-\omega) = \cos x + \omega \sin x - \tfrac{1}{2}\omega^2 \cos x - \tfrac{1}{6}\omega^3 \sin x + \tfrac{1}{24}\omega^4 \cos x + \&c.$$

96. Uſus harum formularum eximius eſt cum in condendis, tum interpolandis tabulis ſinuum & coſinuum. Si enim cogniti fuerint ſinus & coſinus cuiuſpiam arcus x, ex iis facili negotio ſinus & coſinus angulorum $x+\omega$ & $x-\omega$ inveniri poſſunt, ſi quidem differentia ω fuerit ſatis exigua: hoc enim caſu ſeries inventae vehementer convergunt. Ad hoc vero neceſſe eſt, ut arcus ω in partibus radii exprimatur; quod cum arcus 180° ſit: 3, 14159265358979323846 facile fiet: erit enim diviſione per 180 inſtituta

arcus $1^\circ = 0{,}017453292519943295769$
arcus $1' = 0{,}000290888208665721596$
arcus $10'' = 0{,}000048481368110953599$

EXEMPLUM I.

Invenire ſinus & coſinus angulorum 45°, 1', *&* 44°, 59', *ex datis ſinu & coſinu anguli* 45°, *quorum uterque eſt*

$$= \frac{1}{\sqrt{2}} = 0{,}7071067811865.$$

Cum igitur ſit:

$$\sin x = \cos x = 0{,}7071067811865$$
$$\text{atque } \omega = 0{,}0002908882086$$

erit ad multiplicationes facilius inſtituendas:

$$2\omega = 0{,}0005817764173$$
$$3\omega = 0{,}0008726646259$$
$$4\omega = 0{,}0011635528346$$
$$5\omega = 0{,}0014544410432$$
$$6\omega = 0{,}0017453292519$$
$$7\omega = 0{,}0020362174605$$
$$8\omega = 0{,}0023271056692$$
$$9\omega = 0{,}0026179938779$$

Ergo $\omega \sin x$ & $\omega \cos x$ hoc modo invenietur:

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	7 .	0,00020362174605
	0 .	. . . .
	7 .	0,00000203621746
	1 .	. 2908882
	0 .	. . .
	6 .	. 174532
	7 .	. 20362
	8 .	. 2327
	1 .	. 29
	1 .	. . 2
	8 .	. . 2
	6 .	. . 0
ω ſin x = ω coſ x =		0,00020568902490
Ergo $\frac{1}{2}\omega$ coſ x =		0,00010284451245
per ω .	1 .	0,00000002908882
	0 .	. . .
	2 .	. 58177
	8 .	. 23271
	4 .	. 1163
	4 .	. 116
	5 .	. . 14
$\frac{1}{2}\omega^2$ coſ x =		0,00000002991625
$\frac{1}{6}\omega^2$ coſ x =		0,00000000997208
per ω .	9 .	0,00000000000261
	9 .	. . 26
	7 .	. . 2
$\frac{1}{6}\omega^3$ coſ x =		0,00000000000290

Ergo ad ſin 45°, 1′, inveniendum:

Ad ſin x =	0,7071067811865
add. ω coſ x =	2056890249
	0,7073124702114
ſubtr. $\frac{1}{2}\omega^2$ ſin x =	. . 299162
	0,7073124402952

$1+xx=\frac{1}{\sin u^2}$, unde ſit $\frac{dy}{dx}=\frac{1}{1+xx}=\sin u^2$. Cum deinde ſit $dx=\frac{-du}{\sin u^2}$, ſeu $du=-dx\sin u^2$, fiet ulteriora differentialia ſumendo:

$$\frac{ddy}{dx}=2du\sin u\cos u=du\sin 2u=-dx\sin u^2.\sin 2u$$

ideoque $\frac{ddy}{1dx^2}=-\sin u^2.\sin 2u.$

$$\frac{d^3y}{2dx^2}=-du\sin u.\cos u.\sin 2u-du\sin u^2\cos 2u=-du\sin u.\sin 3u$$
$$=dx\sin u^3\sin 3u$$

ideoque $\frac{d^3y}{1.2dx^3}=+\sin u^3.\sin 3u$

$$\frac{d^4y}{1.2.3dx^3}=du\sin u^2.(\cos u.\sin 3u+\sin u.\cos 3u)=du\sin u^2.\sin 4u$$
$$=-dx\sin u^4.\sin 4u$$

ideoque $\frac{d^4y}{1.2.3dx^4}=-\sin u^4.\sin 4u$

$$\frac{d^5y}{1.2.3.4dx^4}=-du\sin u^3(\cos u\sin 4u+\sin u.\cos 4u)=-du.\sin u^3.\sin 5u$$
$$=+dx\sin u^5.\sin 5u$$

ideoque $\frac{d^5y}{1.2.3.4dx^5}=+\sin u^5.\sin 5u$

&c.

Ex quibus colligitur fore:

$$A\,tg(x+\omega)=A\,tg\,x+\frac{\omega}{1}\sin u.\sin u-\frac{\omega^2}{2}\sin u^2.\sin 2u+\frac{\omega^3}{3}\sin u^3.\sin 3u$$
$$-\frac{\omega^4}{4}\sin u^4\sin 4u+\frac{\omega^5}{5}\sin u^5.\sin 5u-\frac{\omega^6}{6}\sin u^6\sin 6u+\&c.$$

ubi

ubi cum ſit A tg $x = y$ & A tg $x = 90° - u$, erit $y = 90° - u$.

88. Si ponatur A cot $x = y$ & A cot $(x + \omega) = z$; erit

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{1.2 dx^2} + \frac{\omega^3 d^3 y}{1.2.3 dx^3} + \frac{\omega^4 d^4 y}{1.2.3.4 dx^4} + \&c.$$

Cum autem ſit $dy = \frac{-dx}{1 + xx}$, termini huius ſeriei congruent praeter primum cum ante inventis, exceptis tantum ſignis. Quare ſi ponatur, ut ante A tang $x = 90° - u$, ſeu A cot $x = u$, ut ſit $u = y$; erit:

$$A \cot(x + \omega) = A \cot x - \frac{\omega}{1} \sin u . \sin u + \frac{\omega^2}{2} \sin u^2 . \sin 2u - \frac{\omega^3}{3} \sin u^3 \sin 3u$$

$$+ \frac{\omega^4}{4} \sin u^4 . \sin 4u - \frac{\omega^5}{5} \sin u^5 . \sin 5u + \&c.$$

quae expreſſio immediate ex praecedente ſequitur: quia enim eſt $A \cot(x + \omega) = 90 - A \operatorname{tang}(x + \omega)$
& $A \cot x = 90 - A \operatorname{tang} x$; erit
$A \cot(x + \omega) - A \cot x = - A \operatorname{tang}(x + \omega) + A \operatorname{tang} x$.

89. Ex his expreſſionibus multa egregia corollaria conſequuntur, prout loco x & ω dati valores ſubſtituantur. Sit igitur primum $x = 0$; & cum ſit $u = 90° - A \operatorname{tang} x$ fiet $u = 90°$; atque $\sin u = 1$; $\sin 2u = 0$; $\sin 3u = -1$; $\sin 4u = 0$; $\sin 5u = 1$; $\sin 6u = 0$; $\sin 7u = -1$; &c. unde fiet

$$A \operatorname{tang} \omega = \frac{\omega}{1} - \frac{\omega^3}{3} + \frac{\omega^5}{5} - \frac{\omega^7}{7} + \frac{\omega^9}{9} - \frac{\omega^{11}}{11} + \&c.$$

quae eſt notiſſima ſeries exprimens arcum, cuius tangens eſt $= \omega$. Sit $x = 1$, erit A tang $x = 45°$, ideoque $u = 45°$, hinc

$\sin u = \frac{1}{\sqrt{2}}$; $\sin 2u = 1$; $\sin 3u = \frac{1}{\sqrt{2}}$; $\sin 4u = 0$;

$\sin 5u = -\frac{1}{\sqrt{2}}$; $\sin 6u = -1$; $\sin 7u = -\frac{1}{\sqrt{2}}$; $\sin 8u = 0$

$\sin 9u = \frac{1}{\sqrt{2}}$ &c. Ex quibus fit:

$$A\,tg(1+\omega)=45^\circ+\frac{\omega}{2}-\frac{\omega^2}{2.2}+\frac{\omega^3}{3.4}-\frac{\omega^5}{5.8}+\frac{\omega^6}{6.8}-\frac{\omega^7}{7.16}$$
$$+\frac{\omega^9}{9.32}-\frac{\omega^{10}}{10.32}+\frac{\omega^{11}}{11.64}-\frac{\omega^{13}}{13.128}+\frac{\omega^{14}}{14.128}-\&c.$$

Si igitur ſit $\omega=-1$; ob $A\,tang(1+\omega)=0$, & $45^\circ=\frac{\pi}{4}$ fiet:

$$\frac{\pi}{4}=\frac{1}{1.2}+\frac{1}{2.2}+\frac{1}{3.2^2}-\frac{1}{5.2^3}-\frac{1}{6.2^3}-\frac{1}{7.2^4}+\frac{1}{9.2^5}+\frac{1}{10.2^5}+\frac{1}{11.2^6}-\&c.$$

qui valor ſi loco arcus 45° ſubſtituatur in illa expreſſione erit: $A\,tang(1+\omega)=$

$$\frac{\omega+1}{1.2}-\frac{\omega^2+1}{2.2}+\frac{\omega^3+1}{3.2^2}-\frac{\omega^5-1}{5.2^3}+\frac{\omega^6-1}{6.2^3}-\frac{\omega^7-1}{7.2^4}+\&c.$$

Illa autem ſeries maxime eſt idonea ad valorem ipſius $\frac{\pi}{4}$ proxime inveniendum Cum ſit

$$\frac{\pi}{4}=\frac{1}{1.2}+\frac{1}{2.2}+\frac{1}{3.2^2}-\frac{1}{5.2^3}-\frac{1}{6.2^3}-\frac{1}{7.2^4}+\&c.$$

termini autem in denominatoribus habentes 2, 6, 10, &c. $\frac{1}{2.2}-\frac{1}{6.2^3}+\frac{1}{10.2^5}-\frac{1}{14.2^7}+\&c.$ exprimunt $\frac{1}{2}A\,tg\frac{1}{2}$; erit

$$\frac{\pi}{4}=\frac{1}{2}A\,tang\frac{1}{2}+\frac{1}{1.2}+\frac{1}{3.2^2}-\frac{1}{5.2^3}-\frac{1}{7.2^4}+\frac{1}{9.2^5}+\frac{1}{11.2^6}\ \&c.$$

In altera autem formula poſito ω negativo, cum ſit

$$A\,tang(1-\omega)=\begin{array}{l}+\frac{1}{1.2}+\frac{1}{2.2}+\frac{1}{3.2^2}-\frac{1}{5.2^3}-\frac{1}{6.2^3}-\frac{1}{7.2^4}+\&c.\\ -\frac{\omega}{1.2}-\frac{\omega^2}{2.2}-\frac{\omega^3}{3.2^2}+\frac{\omega^5}{5.2^3}+\frac{\omega^6}{6.2^3}+\frac{\omega^7}{7.2^4}-\&c.\end{array}$$

ſi fiat $\omega=\frac{1}{2}$; erit:

A

$$A\,\text{tang}\,\tfrac{1}{2} = \begin{cases} \dfrac{1}{1.2} + \dfrac{1}{2.2} + \dfrac{1}{3.2^2} - \dfrac{1}{5.2^3} - \dfrac{1}{6.2^3} - \dfrac{1}{7.2^4} + \&c. \\ -\dfrac{1}{1.2^2} - \dfrac{1}{2.2^3} - \dfrac{1}{3.2^5} + \dfrac{1}{5.2^8} + \dfrac{1}{6.2^9} + \dfrac{1}{7.2^{11}} - \&c. \end{cases}$$

& terminis per 2, 6, 10, &c. diviſis ſeorſim ſumtis erit

$$A\,\text{tg}\,\tfrac{1}{2} = \tfrac{1}{2}A\,\text{tg}\,\tfrac{1}{2} + \frac{1}{1.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} - \frac{1}{7.2^4} + \frac{1}{9.2^5} + \&c.$$

$$-\tfrac{1}{2}A\,\text{tang}\,\tfrac{1}{8} - \frac{1}{1.2^2} - \frac{1}{3.2^5} + \frac{1}{5.2^8} + \frac{1}{7.2^{11}} - \frac{1}{9.2^{14}} - \&c.$$

ideoque $\tfrac{1}{2}A\,\text{tang}\,\tfrac{1}{2} = \dfrac{1}{1.2} + \dfrac{1}{3.2^2} - \dfrac{1}{5.2^3} - \dfrac{1}{7.2^4} + \&c.$

$$-\tfrac{1}{2}A\,\text{tang}\,\tfrac{1}{8} - \frac{1}{1.2^2} - \frac{1}{3.2^5} + \frac{1}{5.2^8} + \frac{1}{7.2^{11}} - \&c.$$

qui valor ſi in ſuperiore ſerie ſubſtituatur, atque A tang $\tfrac{1}{8}$ ipſe in ſeriem convertatur, reperietur

$$\frac{\pi}{4} = \begin{cases} 1 + \dfrac{1}{3.2^1} - \dfrac{1}{5.2^2} - \dfrac{1}{7.2^3} + \dfrac{1}{9.2^4} + \&c. \\ -\dfrac{1}{1.2^2} - \dfrac{1}{3.2^5} + \dfrac{1}{5.2^8} + \dfrac{1}{7.2^{11}} - \dfrac{1}{9.2^{14}} - \&c. \\ -\dfrac{1}{1.2^4} + \dfrac{1}{3.2^{10}} - \dfrac{1}{5.2^{16}} + \dfrac{1}{7.2^{22}} - \dfrac{1}{9.2^{28}} + \&c. \end{cases}$$

90. Sequuntur hae multaeque aliae ſeries ex poſitione $x = 1$: ſin autem ponamus $x = \sqrt{3}$, ut ſit $A\,\text{tang}\,x = 60^\circ$, fiet $u = 30$, & $\text{ſin}\,u = \tfrac{1}{2}$; $\text{ſin}\,2u = \dfrac{\sqrt{3}}{2}$; $\text{ſin}\,3u = 1$; $\text{ſin}\,4u = \dfrac{\sqrt{3}}{2}$; $\text{ſin}\,5u = \tfrac{1}{2}$; $\text{ſin}\,6u = 0$; $\text{ſin}\,7u = -\tfrac{1}{2}$; &c. unde erit:

$$A\,\text{tang}(\sqrt{3} + \omega) = 60^\circ + \frac{\omega}{1.2^2} - \frac{\omega^2\sqrt{3}}{2.2^3} + \frac{\omega^3}{3.2^3} - \frac{\omega^4\sqrt{3}}{4.2^5}$$

$$+\frac{\omega^5}{5.2^6}-\frac{\omega^7}{7.2^8}+\frac{\omega^8\sqrt{3}}{8.2^9}-\frac{\omega^9}{9.2^9}+\frac{\omega^{10}\sqrt{3}}{10.2^{11}}-\frac{\omega^{11}}{11.2^{12}}+\text{\&c.}$$

Sin autem ponatur $x=\frac{1}{\sqrt{3}}$, ut ſit A tang $x=30^\circ$; erit $u=60^\circ$; atque $\text{ſin}\,u=\frac{\sqrt{3}}{2}$; $\text{ſin}\,2u=\frac{\sqrt{3}}{2}$; $\text{ſin}\,3u=0$; $\text{ſin}\,4u=-\frac{\sqrt{3}}{2}$; $\text{ſin}\,5u=-\frac{\sqrt{3}}{2}$; $\text{ſin}\,6u=0$; $\text{ſin}\,7u=\frac{\sqrt{3}}{2}$; &c.

quibus valoribus ſubſtitutis erit:

$$\text{A tg}\left(\frac{1}{\sqrt{3}}+\omega\right)=30^\circ+\frac{3\omega}{1.2^2}-\frac{3\omega^2\sqrt{3}}{2.2^3}+\frac{3^2\omega^4\sqrt{3}}{4.2^5}-\frac{3^3\omega^5}{5.2^6}+\text{\&c.}$$

ſi igitur ſit $\omega=-\frac{1}{\sqrt{3}}$, ob $30^\circ=\frac{\pi}{6}$; erit:

$$\frac{\pi}{6\sqrt{3}}=\frac{1}{1.2^2}+\frac{1}{2.2^3}-\frac{1}{4.2^5}-\frac{1}{5.2^6}+\frac{1}{7.2^8}+\frac{1}{8.2^9}-\text{\&c.}$$

91. Reſumamus expreſſionem generalem inventam:

$$\text{A tang.}\,(x+\omega)=\text{A tang}\,x$$

$$+\frac{\omega}{1}\,\text{ſin}\,u.\,\text{ſin}\,u-\frac{\omega^2}{2}\,\text{ſin}\,u^2.\,\text{ſin}\,2u+\frac{\omega^3}{3}\,\text{ſin}\,u^3.\,\text{ſin}\,3u-\text{\&c.}$$

ac ponamus $\omega=-x$, ut ſit $\text{A tang}(x+\omega)=0$, eritque

$$\text{A tang}\,x=$$

$$\frac{x}{1}\,\text{ſin}\,u.\,\text{ſin}\,u+\frac{x^2}{2}\,\text{ſin}\,u^2.\,\text{ſin}\,2u+\frac{x^3}{3}\,\text{ſin}\,u^3.\,\text{ſin}\,3u+\text{\&c.}$$

Cum autem ſit $\text{A tang}\,x=90-u=\frac{\pi}{2}-u$;

erit: $x=\cot u=\frac{\text{coſ}\,u}{\text{ſin}\,u}$. Quamobrem erit:

$$\frac{\pi}{2}=u+\text{coſ}\,u.\,\text{ſin}\,u+\tfrac{1}{2}\,\text{coſ}\,u^2.\,\text{ſin}\,2u+\tfrac{1}{3}\,\text{coſ}\,u^3.\,\text{ſin}\,3u+\tfrac{1}{4}\,\text{coſ}\,u^4.\,\text{ſin}\,4u+\text{\&c.}$$

quae

quae ſeries eo magis eſt notatu digna, quod quicunque arcus loco u accipiatur, valor ſeriei ſemper prodeat idem $= \frac{\pi}{2}$. Sin autem ſit $\omega = -2x$, ob $A\,\mathrm{tang}(-x) = -A\,\mathrm{tg}\,x$; fiet:

$$2\,A\,\mathrm{tang}\,x =$$
$$\frac{2x}{1}\sin u.\sin u + \frac{4x^2}{2}\sin u^2.\sin 2u + \frac{8x^3}{3}\sin u^3.\sin 3u + \&c.$$

Cum autem ſit $A\,\mathrm{tang}\,x = \frac{\pi}{2} - u$ & $x = \frac{\cos u}{\sin u}$, erit:

$$\pi = 2u + \frac{2}{1}\cos u.\sin u + \frac{2^2}{2}\cos u^2.\sin 2u + \frac{2^3}{3}\cos u^3.\sin 3u + \&c.$$

Sit $u = 45^\circ = \frac{\pi}{4}$; erit $\cos u = \frac{1}{\sqrt{2}}$; $\sin u = \frac{1}{\sqrt{2}}$; $\sin 2u = 1$; $\sin 3u = \frac{1}{\sqrt{2}}$; $\sin 4u = 0$; $\sin 5u = \frac{-1}{\sqrt{2}}$; $\sin 6u = -1$; $\sin 7u = \frac{-1}{\sqrt{2}}$; $\sin 8u = 0$; $\sin 9u = \frac{1}{\sqrt{2}}$; eritque

$$\frac{\pi}{2} = \frac{1}{1} + \frac{2}{2} + \frac{2}{3} - \frac{2^2}{5} - \frac{2^3}{6} - \frac{2^3}{7} + \frac{2^4}{9} + \frac{2^5}{10} + \frac{2^5}{11} - \&c.$$

quae ſeries etſi divergit, tamen ob ſimplicitatem eſt notatu digna.

92. Ponatur in expreſſione generali inventa:

$$\omega = -x - \frac{1}{x} = \frac{-1}{\sin u.\cos u}, \quad \text{ob} \quad x = \frac{\cos u}{\sin u}; \quad \text{erit:}$$

$$A\,\mathrm{tang}(x + \omega) = A\,\mathrm{tang} - \frac{1}{x} = - A\,\mathrm{tang}\frac{1}{x} = -\frac{\pi}{2} + A\,\mathrm{tang}\,x.$$

Hinc ergo obtinebitur ſequens expreſſio:

$$\frac{\pi}{2} = \frac{\sin u}{1\cos u} + \frac{\sin 2u}{2\cos u^2} + \frac{\sin 3u}{3\cos u^3} + \frac{\sin 4u}{4\cos u^4} + \frac{\sin 5u}{5\cos u^5} + \&c.$$

quae poſito $u = 45^\circ$ dat eandem ſeriem, quam ultimo loco invenimus. Sin autem ponamus $\omega = -\sqrt{(1 + xx)}$

ob

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$$\text{ob} \quad x = \frac{\operatorname{coſ} u}{\operatorname{ſin} u}, \quad \text{fiet} \quad \omega = -\frac{1}{\operatorname{ſin} u}, \quad \&$$

$$\text{A tang}\,[x - \surd(1+xx)] - \text{A tang}\,[\surd(1+xx) - x]$$

$$= -\tfrac{1}{2}\,\text{A tang}\,\frac{1}{x} = -\tfrac{1}{2}\left(\frac{\pi}{2} - \text{A tang}\,x\right) = -\tfrac{1}{2}\,u,$$

$$\& \quad \text{A tang}\,x = \frac{\pi}{2} - u. \qquad \text{Hancobrem erit:}$$

$$\frac{\pi}{2} = \tfrac{1}{2}u + \tfrac{1}{1}\operatorname{ſin} u + \tfrac{1}{2}\operatorname{ſin} 2u + \tfrac{1}{3}\operatorname{ſin} 3u + \tfrac{1}{4}\operatorname{ſin} 4u + \&c.$$

Quodſi haec aequatio differentietur erit:

$$0 = \tfrac{1}{2} + \operatorname{coſ} u + \operatorname{coſ} 2u + \operatorname{coſ} 3u + \operatorname{coſ} 4u + \operatorname{coſ} 5u + \&c.$$

cuius ratio ex natura ſerierum recurrentium intelligitur.

93. Si ſimili modo ſeries ante inventae differentientur, novae ſeries ſummabiles reperientur. Ac primo quidem ex ſerie:

$$\text{A tang}\,(1+\omega) = \frac{\pi}{4} + \frac{\omega}{2} - \frac{\omega^2}{2.2} + \frac{\omega^3}{3.4} - \frac{\omega^5}{5.8} + \frac{\omega^6}{6.8} - \&c.$$

ſequitur

$$\frac{1}{2+2\omega+\omega^2} = \tfrac{1}{2} - \frac{\omega}{2} + \frac{\omega^2}{4} - \frac{\omega^4}{8} + \frac{\omega^5}{8} - \frac{\omega^6}{16} + \frac{\omega^8}{32} - \&c.$$

quae oritur ex evolutione fractionis $\frac{2-2\omega+\omega^2}{4+\omega^4} = \frac{1}{2+2\omega+\omega^2}$,

Deinde iſta ſeries:

$$\frac{\pi}{2} = u + \operatorname{coſ} u \operatorname{ſin} u + \tfrac{1}{2}\operatorname{coſ} u^2.\operatorname{ſin} 2u + \tfrac{1}{3}\operatorname{coſ} u^3 \operatorname{ſin} 3u + \tfrac{1}{4}\operatorname{coſ} u^4 \operatorname{ſin} 4u + \&c.$$

per differentiationem dabit:

$$0 = 1 + \operatorname{coſ} 2u + \operatorname{coſ} u.\operatorname{coſ} 3u + \operatorname{coſ} u^2.\operatorname{coſ} 4u + \operatorname{coſ} u^3 \operatorname{coſ} 5u + \&c.$$

Denique ſeries $\frac{\pi}{2} = \frac{\operatorname{ſin} u}{\operatorname{coſ} u} + \frac{\operatorname{ſin} 2u}{2\operatorname{coſ} u^2} + \frac{\operatorname{ſin} 3u}{3\operatorname{coſ} u^3} + \frac{\operatorname{ſin} 4u}{4\operatorname{coſ} u^4} + \&c.$

$$\text{dat} \quad 0 = \frac{1}{\operatorname{coſ} u^2} + \frac{\operatorname{coſ} u}{\operatorname{coſ} u^3} + \frac{\operatorname{coſ} 2u}{\operatorname{coſ} u^4} + \frac{\operatorname{coſ} 3u}{\operatorname{coſ} u^5} + \frac{\operatorname{coſ} 4u}{\operatorname{coſ} u^6} + \&c.$$

ſeu

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ſeu $0 = 1 + \frac{\cos u}{\cos u} + \frac{\cos 2u}{\cos u^2} + \frac{\cos 3u}{\cos u^3} + \frac{\cos 4u}{\cos u^4} + \&c.$

94. Imprimis autem expreſſio inventa:

$$A\,\text{tang}\,(x+\omega) =$$

$$A\,\text{tang}\,x + \frac{\omega}{1}\sin u.\sin u - \frac{\omega^2}{2}\sin u^2.\sin 2u + \frac{\omega^3}{3}\sin u^3.\sin 3u - \&c.$$

exiſtente $x = \cot u$ ſeu $u = A\cot x = 90° - A\,\text{tang}\,x$ inſerviet ad angulum ſeu arcum datae cuique tangenti reſpondentem inveniendum. Sit enim propoſita tangens $= t$, quaeraturque in tabulis tangens ad hanc proxime accedens $= x$, cui reſpondeat arcus $= y$; eritque $u = 90° - y$. Tum ponatur $x + \omega = t$, ſeu $\omega = t - x$; eritque arcus quaeſitus:

$$= y + \frac{\omega}{1}\sin u.\sin u - \frac{\omega^2}{2}\sin u^2.\sin 2u + \&c.$$

quae regula tum praecipue eſt utilis, cum tangens propoſita fuerit admodum magna, ac propterea arcus quae ſitus parum a $90°$ diſcrepet. His enim caſibus ob tangentes vehementer increſcentes, ſolita methodus interpolationum nimium a veritate abducit. Sit ergo propoſitum hoc exemplum.

EXEMPLUM.

Quaeratur arcus, cuius tangens ſit $= 100$, *poſito radio* $= 1$.

Arcus proxime quaeſito aequalis eſt $89°, 25'$, cuius tangens eſt $x = 98, 217943$ ſecund.

quae ſubtrahatur a $t = 100, 000000$

remanebit $\omega = 1, 782057$

Deinde cum ſit $y = 89°, 25'$, erit $u = 0°, 35'$, $2u = 1°, 10'$, $3u = 1°; 45'$, &c. Iam ſinguli termini per logarithmos inveſtigentur.

Ad

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Ad	$l\omega =$	0,2509215	
add.	$l\,\text{ſin}\,u =$	8,0077867	
	$l\,\text{ſin}\,u =$	8,0077867	
	$l\omega\,\text{ſin}\,u.\,\text{ſin}\,u =$	6,2664949	
		4,6855749	
ſubtr.	$=$	1,5809200	
Ergo	$\omega\,\text{ſin}\,u.\,\text{ſin}\,u =$	38,09956	ſecund.
Ad	$l\omega\,\text{ſin}\,u^2 =$	6,2664949	
add.	$l\omega =$	0,2509215	
	$l\,\text{ſin}\,2u =$	8,3087941	
		4,8262105	
ſubtr.	$l\,2 =$	0,3010300	
	$l\,\tfrac{1}{2}\omega^2\,\text{ſin}\,u^2.\,\text{ſin}\,2u =$	4,5251805	
ſubtr.		4,6855749	
Remanet		9,8396056	
Ergo	$\tfrac{1}{2}\omega^2\,\text{ſin}\,u^2.\,\text{ſin}\,2u =$	0,69120	ſecund.
Porro ad	$l\omega^3 =$	0,7527645	
add.	$l\,\text{ſin}\,u^3 =$	4,0233601	
	$l\,\text{ſin}\,3u =$	8,4848479	
		3,2609725	
ſubtr.	$l\,3 =$	0,4771213	
		2,7838512	
ſubtr.		4,6855749	
		8,0982763	
Ergo	$\tfrac{1}{3}\omega^3\,\text{ſin}\,u^3\,\text{ſin}\,3u =$	0,01254	ſecund.
Denique ad	$l\omega^4 =$	1,0036860	
add.	$l\,\text{ſin}\,u^4 =$	2,0311468	
	$l\,\text{ſin}\,4u =$	8,6097341	
		1,6445669	
ſubtr.	$l\,4 =$	0,6020600	
		1,0425069	
ſubtr.		4,6855749	
		6,3569320	
Ergo	$\tfrac{1}{4}\omega^4\,\text{ſin}\,u^4\,\text{ſin}\,4u =$	0,00023	ſecund.

Hinc:

Termini addendi	Termini ſubtrahendi
38,09956	0,69120
0,01254	0,00023
38,11210	0,69143
ſubtr. 0,69143	

$37,42067 = 37^{\mathrm{II}}, 25^{\mathrm{III}}, 14^{\mathrm{IV}}, 24^{\mathrm{V}}, 36^{\mathrm{VI}}.$

Quocirca arcus, cuius tangens centies ſuperat radium erit:

$89^{\circ}, 25^{\mathrm{I}}, 37^{\mathrm{II}}, 25^{\mathrm{III}}, 14^{\mathrm{IV}}, 24^{\mathrm{V}}, 36^{\mathrm{VI}},$

neque error ad minuta quarta aſcendit; ſed in minutis tantum quintis ineſſe poteſt, ex quo vere hunc angulum pronunciare poterimus $= 89^{\circ}, 25^{\mathrm{I}}, 37^{\mathrm{II}}, 25^{\mathrm{III}}, 14^{\mathrm{IV}}$. Si tangens adhuc maior proponatur, etiamſi fortaſſe ω maius prodeat, tamen ob u angulum adhuc minorem, aeque expedite arcus definiri poterit.

95. Cum hic pro y arcum circuli ſubſtituerimus, nunc functiones reciprocas in locum y ponamus, cuiuſmodi ſunt $\sin x$, $\cos x$, $\mathrm{tang}\, x$, $\cot x$, &c. Sit igitur $y = \sin x$, poſitoque $x + \omega$ loco x, fiet: $z = \sin(x + \omega)$, atque aequatio

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \&c.$$

ob $\frac{dy}{dx} = \cos x$; $\frac{ddy}{dx^2} = -\sin x$; $\frac{d^3y}{dx^3} = -\cos x$; &c. dabit

$$\sin(x + \omega) = \sin x + \omega \cos x - \tfrac{1}{2}\omega^2 \sin x - \tfrac{1}{6}\omega^3 \cos x + \tfrac{1}{24}\omega^4 \sin x + \&c.$$

& ſumto ω negativo erit:

$$\sin(x - \omega) = \sin x - \omega \cos x - \tfrac{1}{2}\omega^2 \sin x + \tfrac{1}{6}\omega^3 \cos x + \tfrac{1}{24}\omega^4 \sin x - \&c.$$

Quod ſi vero ſtatuatur $y = \cos x$,

ob $\frac{dy}{dx} = -\sin x$; $\frac{ddy}{dx^2} = -\cos x$; $\frac{d^3y}{dx^3} = \sin x$; $\frac{d^4y}{dx^4} = \mathrm{cf}\, x$; &c.

erit:

$$\cos(x + \omega) = \cos x - \omega \sin x - \tfrac{1}{2}\omega^2 \cos x + \tfrac{1}{6}\omega^3 \sin x + \tfrac{1}{24}\omega^4 \cos x - \&c.$$

& facto ω negativo erit:

$$\cos(x-\omega)=\cos x+\omega\sin x-\tfrac{1}{2}\omega^2\cos x-\tfrac{1}{6}\omega^3\sin x+\tfrac{1}{24}\omega^4\cos x+\&c.$$

96. Uſus harum formularum eximius eſt cum in condendis, tum interpolandis tabulis ſinuum & coſinuum. Si enim cogniti fuerint ſinus & coſinus cuiuſpiam arcus x, ex iis facili negotio ſinus & coſinus angulorum $x+\omega$ & $x-\omega$ inveniri poſſunt, ſi quidem differentia ω fuerit ſatis exigua: hoc enim caſu ſeries inventae vehementer convergunt. Ad hoc vero neceſſe eſt, ut arcus ω in partibus radii exprimatur; quod cum arcus 180° ſit: 3, 14159265358979323846 facile fiet: erit enim diviſione per 180 inſtituta

arcus $1^\circ = 0{,}01745329251994329576\,9$

arcus $1' = 0{,}00029088820866572159\,6$

arcus $10'' = 0{,}00004848136811095359\,9$

EXEMPLUM I.

Invenire ſinus & coſinus angulorum 45°, 1', *&* 44°, 59', *ex datis ſinu & coſinu anguli* 45°, *quorum uterque eſt*

$$=\frac{1}{\sqrt{2}}=0{,}7071067811865.$$

Cum igitur ſit:

$$\sin x=\cos x=0{,}7071067811865$$

$$\text{atque}\quad \omega=0{,}0002908882086$$

erit ad multiplicationes facilius inſtituendas:

$$2\omega=0{,}0005817764173$$
$$3\omega=0{,}0008726646259$$
$$4\omega=0{,}0011635528346$$
$$5\omega=0{,}0014544410432$$
$$6\omega=0{,}0017453292519$$
$$7\omega=0{,}0020362174605$$
$$8\omega=0{,}0023271056692$$
$$9\omega=0{,}0026179938779$$

Ergo $\omega\sin x$ & $\omega\cos x$ hoc modo invenietur:

7 .	0,00020362174605
0 .	. . . .
7 .	0,00000203621746
1 .	. 2908882
0 .	. . .
6 .	. 174532
7 .	. 20362
8 .	. 2327
1 .	. 29
1 .	. . 2
8 .	. . 2
6 .	. . 0

$\omega\,\text{ſin}\,x = \omega\,\text{coſ}\,x = 0,00020568902490$

Ergo $\tfrac{1}{2}\omega\,\text{coſ}\,x = 0,00010284451245$

per ω . 1 .	0,0000000290882
0 .	. . .
2 .	. 58177
8 .	. 23271
4 .	. 1163
4 .	. 116
5 .	. . 14

$\tfrac{1}{2}\omega^2\,\text{coſ}\,x = 0,0000000 2991625$

$\tfrac{1}{6}\omega^2\,\text{coſ}\,x = 0,00000000997208$

per ω . 9 .	0,00000000000261
9 .	. . 26
7 .	. . 2

$\tfrac{1}{6}\omega^3\,\text{coſ}\,x = 0,00000000000290$

Ergo ad ſin 45°, 1′, inveniendum:

Ad $\text{ſin}\,x = 0,7071067811865$

add. $\omega\,\text{coſ}\,x =$ 2056890249

0,70731247021114

ſubtr. $\tfrac{1}{2}\omega^2\,\text{ſin}\,x =$ 299162

0,70731244029 52

$$\begin{array}{rl} & 0,7073124402952 \\ \text{ſubtr.}\ \tfrac{1}{6}\omega^3\cos x = & \underline{\qquad\qquad\qquad 29} \\ \sin 45^\circ,\ 1', = & 0,7073124402923 = \cos 44^\circ,\ 59'. \end{array}$$

At ad $\cos 45^\circ,\ 1'$, inveniendum:

$$\begin{array}{rl} A\cos x = & 0,7071067811865 \\ \text{ſubtr.}\ \omega\sin x = & \underline{\qquad 2056890249} \\ & 0,7069010921616 \\ \text{ſubtr.}\ \tfrac{1}{2}\omega^2\cos x = & \underline{\qquad\qquad 299162} \\ & 0,7069010622454 \\ \text{add.}\ \tfrac{1}{6}\omega^3\sin x = & \underline{\qquad\qquad\qquad 29} \\ \cos 45^\circ,\ 1', = & 0,7069010622483 = \sin 44^\circ,\ 59'. \end{array}$$

EXEMPLUM II.

Ex datis ſinu & coſinu arcus $67^\circ,\ 30'$, invenire ſinus & coſinus arcuum $67^\circ,\ 31'$, & $67^\circ,\ 29'$.

Abſolvamus hunc calculum in fractionibus decimalibus, tantum ad 7 notas, uti tabulae vulgares conſtrui ſolent, ſicque negotium facile per logarithmos conficietur. Cum ſit $x = 67^\circ, 30'$, & $\omega = 0,000290888$; erit: $l\omega = 6,4637259$ &

$$\begin{array}{rl} l\sin x = 9,9656153 & ; \quad l\cos x = 9,5828397 \\ \underline{l\omega = 6,4637259} & ; \quad \underline{l\omega = 6,4637259} \\ l\omega\sin x = 6,4293412 & ; \quad l\omega\cos x = 6,0465656 \\ \underline{l\tfrac{1}{2}\omega = 6,1626959} & ; \quad \underline{l\tfrac{1}{2}\omega = 6,1626959} \\ l\tfrac{1}{2}\omega^2\sin x = 2,5920371 & ; \quad l\tfrac{1}{2}\omega^2\cos x = 2,2092615 \end{array}$$

ergo:

$$\begin{array}{rl} \omega\sin x = 0,00026874 & ; \quad \omega\cos x = 0,00011132 \\ \tfrac{1}{2}\omega^2\sin x = 0,00000004 & ; \quad \tfrac{1}{2}\omega^2\cos x = 0,00000001 \end{array}$$

unde fit:

$$\begin{array}{rl} \sin 67^\circ,\ 31' = 0,9239908 & ; \quad \cos 67^\circ,\ 31' = 0,3824147 \\ \sin 67^\circ,\ 29' = 0,9237681 & ; \quad \cos 67^\circ,\ 29' = 0,3829522 \end{array}$$

ubi ne quidem terminis $\tfrac{1}{2}\omega^2\sin x$ & $\tfrac{1}{2}\omega^2\cos x$ erat opus.

97. Ex feriebus quas ſupra invenimus:

$$\sin(x+\omega) = \sin x + \omega\cos x - \tfrac{1}{2}\omega^2\sin x - \tfrac{1}{6}\omega^3\cos x + \tfrac{1}{24}\omega^4\sin x + \&c.$$

coſ

$$\cos(x+\omega) = \cos x - \omega \sin x - \tfrac{1}{2}\omega^2 \cos x + \tfrac{1}{6}\omega^3 \sin x + \tfrac{1}{24}\omega^4 \cos x - \&c.$$
$$\sin(x-\omega) = \sin x - \omega \cos x - \tfrac{1}{2}\omega^2 \sin x + \tfrac{1}{6}\omega^3 \cos x + \tfrac{1}{24}\omega^4 \sin x - \&c.$$
$$\cos(x-\omega) = \cos x + \omega \sin x - \tfrac{1}{2}\omega^2 \cos x - \tfrac{1}{6}\omega^3 \sin x + \tfrac{1}{24}\omega^4 \cos x + \&c.$$

ſequitur per combinationem fore:

$$\frac{\sin(x+\omega) + \sin(x-\omega)}{2} =$$
$$\sin x - \tfrac{1}{2}\omega^2 \sin x + \tfrac{1}{24}\omega^4 \sin x - \tfrac{1}{720}\omega^6 \sin x + \&c. = \sin x \cos \omega$$

Et

$$\frac{\sin(x+\omega) - \sin(x-\omega)}{2} =$$
$$\omega \cos x - \tfrac{1}{6}\omega^3 \cos x + \tfrac{1}{120}\omega^5 \cos x - \&c. = \cos x \sin \omega$$

unde prodeunt ſeries pro ſinibus & coſinibus iam ſupra inventae:

$$\cos \omega = 1 - \tfrac{1}{2}\omega^2 + \tfrac{1}{24}\omega^4 - \tfrac{1}{720}\omega^6 + \&c.$$
$$\sin \omega = \omega - \tfrac{1}{6}\omega^3 + \tfrac{1}{120}\omega^5 - \tfrac{1}{5040}\omega^7 + \&c.$$

quae eaedem ſeries ex primis ponendo $x = 0$ conſequuntur; cum enim ſit $\cos x = 1$ & $\sin x = 0$ prima ſeries $\sin \omega$, ſecunda vero $\cos \omega$ exhibebit.

98. Ponamus nunc quoque $y = \tang x$, ut ſit $z = \tang(x+\omega)$, erit ob

$$y = \frac{\sin x}{\cos x}; \quad \frac{dy}{dx} = \frac{1}{\cos x^2}; \quad \frac{ddy}{2dx^2} = \frac{\sin x}{\cos x^3};$$
$$\frac{d^3y}{2dx^3} = \frac{1}{\cos x^2} + \frac{3\sin x^2}{\cos x^4} = \frac{3}{\cos x^4} - \frac{2}{\cos x^2};$$
$$\frac{d^4y}{2.4dx^4} = \frac{3\sin x}{\cos x^5} - \frac{\sin x}{\cos x^3};$$
$$\frac{d^5y}{2.4dx^5} = \frac{15}{\cos x^6} - \frac{15}{\cos x^4} + \frac{2}{\cos x^2};$$

unde ſequitur fore:

$$\tang(x+\omega) = \tang x + \frac{\omega}{\cos x^2} + \frac{\omega^2 \sin x}{\cos x^3} + \frac{\omega^3}{\cos x^4} + \frac{\omega^4 \sin x}{\cos x^5}$$
$$- \frac{2\omega^3}{3\cos x^2} - \frac{\omega^4 \sin x}{3\cos x^3};$$

cu-

cuius formulae ope ex data cuiuſvis anguli tangente inveniri poſſunt tangentes angulorum proximorum. Quia vero ſuperior ſeries eſt geometrica, ea in unam ſummam collecta erit:

$$\tang(x+\omega) = \tang x + \frac{\omega + \omega^2 \tang x}{\cos x^2 - \omega^2} - \frac{2\omega^3}{3\cos x^2} - \frac{\omega^4 \sin x}{3\cos x^3} \text{ \&c.} \quad \text{ſeu}$$

$$\tang(x+\omega) = \frac{\sin x \cos x + \omega}{\cos x^2 - \omega^2} - \frac{2\omega^3}{3\cos x^2} - \frac{\omega^4 \sin x}{3\cos x^3} \text{ \&c.}$$

quae formula in hunc finem commodius adhibetur.

99. Similes expreſſiones quoque pro logarithmis ſinuum, coſinuum & tangentium inveniri poſſunt. Sit enim $y =$ logarithmo ſinus anguli x, quod ita exprimamus $y = l\sin x$, & $z = l\sin(x+\omega)$, ob $\frac{dy}{dx} = \frac{n\cos x}{\sin x}$; erit: $\frac{ddy}{dx^2} = \frac{-n}{\sin x^2}$; $\frac{d^3y}{dx^3} = \frac{+2n\cos x}{\sin x^3}$ &c. unde fiet:

$$z = l\sin(x+\omega) = l\sin x + \frac{n\omega\cos x}{\sin x} - \frac{n\omega^2}{2\sin x^2} + \frac{n\omega^3\cos x}{3\sin x^3} \text{ \&c.}$$

ubi n denotat numerum, per quem logarithmi hyperbolici multiplicari debent, ut prodeant logarithmi propoſiti. Sin autem ſit

$$y = l\tang x \quad \& \quad z = l\tang(x+\omega) \quad \text{fiet:}$$

$$\frac{dy}{dx} = \frac{n}{\sin x\cos x} = \frac{2n}{\sin 2x}; \quad \frac{ddy}{2dx^2} = \frac{-2n\cos 2x}{(\sin 2x)^2}; \quad \text{ideoque}$$

$$l\tang(x+\omega) = l\tang x + \frac{2n\omega}{\sin 2x} - \frac{n\omega^2\cos 2x}{(\sin 2x)^2} \text{ \&c.}$$

quarum formularum ope logarithmi ſinuum & tangentium interpolari poſſunt.

100. Ponamus denotare y arcum cuius ſinus logarithmus ſit $= x$, ſeu ut ſit $y = A.l\sin x$, & z eſſe arcum, cuius ſinus logarithmus ſit $= x + \omega$, ſeu $z = A.l\sin(x+\omega)$; erit $x = l\sin y$, & $\frac{dx}{dy} = \frac{n\cos y}{\sin y}$, unde $\frac{dy}{dx} = \frac{\sin y}{n\cos y}$; erit:

$$\frac{ddy}{dx} = \frac{dy}{n\cos y^2} = \frac{dx\sin y}{n^2\cos y^3}; \quad \text{ergo} \quad \frac{ddy}{dx^2} = \frac{\sin y}{n^2\cos y^3}:$$

Conſequenter $z = y + \frac{\omega \sin y}{n \cos y} + \frac{\omega^2 \sin y}{2n^2 \cos y^3} + \&c.$

Simili modo ſi logarithmus coſinus detur, expreſſio reperietur. Siu autem ſit

$$y = A.l\,\mathrm{tang}\,x \quad \& \quad z = A.l\,\mathrm{tang}\,(x + \omega).$$

Cum ſit $x = l\,\mathrm{tang}\,y$; fiet:

$$\frac{dx}{dy} = \frac{n}{\sin y \cos y}, \quad \& \quad \frac{dy}{dx} = \frac{\sin y \cos y}{n} = \frac{\sin 2y}{2n};$$

quare $\frac{ddy}{dx} = \frac{2dy \cos 2y}{2n} = \frac{dx \sin 2y \cos 2y}{2nn}$ &

$$\frac{ddy}{dx^2} = \frac{\sin 2y \cos 2y}{2nn} = \frac{\sin 4y}{4nn}; \quad \frac{d^3y}{dx^3} = \frac{\sin 2y . \cos 4y}{2n^3} \&c. \quad \text{hinc}$$

$$z = y + \frac{\omega \sin 2y}{2n} + \frac{\omega^2 \sin 2y \cos 2y}{4nn} + \frac{\omega^3 \sin 2y . \cos 4y}{12n^3} + \&c.$$

101. Quoniam uſus harum expreſſionum in condendis tabulis logarithmorum ſinuum & tangentium ex antecedentibus facile perſpici poteſt, his diutius non immorabimur. Conſideremus ergo adhuc huiuſmodi valorem:

$y = e^x \sin nx$; ſitque $z = e^{x+\omega} \sin n(x+\omega)$: quia eſt

$$\frac{dy}{dx} = e^x (\sin nx + n \cos nx)$$

$$\frac{ddy}{dx^2} = e^x [(1 - nn) \sin nx + 2n \cos nx]$$

$$\frac{d^3y}{dx^3} = e^x [(1 - 3nn) \sin nx + n(3 - nn) \cos nx]$$

$$\frac{d^4y}{dx^4} = e^x [(1 - 6mm + n^4) \sin nx + n(4 - 4nn) \cos nx]$$

$$\frac{d^5y}{dx^5} = e^x [(1 - 10nn + 5n^4) \sin nx + n(5 - 10nn + n^4) \cos nx]$$

His ſubſtitutis & diviſione per e^x inſtituta erit: $e^\omega \sin n(x+\omega) =$

$$\sin nx + \omega \sin nx + \frac{(1 - nn)}{2} \omega^2 \sin nx + n\omega \cos nx + \frac{2n\omega^2}{2} \cos nx$$

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$$+\frac{(1-3nn)}{6}\omega^3 \sin nx+\frac{(1-6nn+n^4)}{24}\omega^4 \sin nx+\&c.$$

$$+\frac{n(3-nn)}{6}\omega^3 \cos nx+\frac{n(4-4nn)}{24}\omega^4 \cos nx+\&c.$$

102. Hinc plurima egregia corollaria deduci poſſunt; ſufficiat autem nobis haec annotaſſe:

Si fuerit $x=0$ erit:

$$e^{\omega}\sin n\omega = n\omega$$

$$+\frac{2n\omega^2}{2}+\frac{n(3-nn)}{6}\omega^3+\frac{n(4-4nn)}{24}\omega^4+\frac{n(5-10n^2+n^4)}{120}\omega^5+\&c.$$

Si ſit $\omega=-x$, ob $\sin n(x+\omega)=0$; erit:

$$\tang nx = \frac{nx-\frac{2n}{2}x^2+\frac{n(3-nn)}{6}x^3-\frac{n(4-4nn)}{24}x^4+\frac{n(5-10n^2+n^4)}{120}x^5}{1-x+\frac{(1-nn)}{2}x^2-\frac{(1-3nn)}{6}x^3+\frac{(1-6nn+n^4)}{24}x^4-\&c.}$$

Generaliter vero ſi ſit $n=1$ habebitur:

$$e^{\omega}\sin(x+\omega)=\sin x(1+\omega-\tfrac{1}{3}\omega^3-\tfrac{1}{6}\omega^4-\tfrac{1}{30}\omega^5+\tfrac{1}{630}\omega^7+\&c.)$$
$$+\omega\cos x(1+\omega+\tfrac{1}{3}\omega^2-\tfrac{1}{30}\omega^4-\tfrac{1}{90}\omega^5-\tfrac{1}{630}\omega^6+\&c.)$$

Sin autem ſit $n=0$, ob $\sin n(x+\omega)=n(x+\omega)$, & $\sin nx=nx$ atque $\cos nx=1$, ſi ubique per n dividatur, prodibit:

$$e^{\omega}(x+\omega)=x+\omega x+\tfrac{1}{2}\omega^2 x+\tfrac{1}{6}\omega^3 x+\tfrac{1}{24}\omega^4 x+\&c.$$
$$+\omega+\omega^2+\tfrac{1}{2}\omega^3+\tfrac{1}{6}\omega^4+\tfrac{1}{24}\omega^5+\&c.$$

cuius ſeriei ratio eſt manifeſta.

CA-

CAPUT V.

INVESTIGATIO SUMMAE SERIERUM EX TERMINO GENERALI.

103.

Sit Seriei cuiuſque terminus generalis $= y$, reſpondens indici x, ita ut y ſit functio quaecunque ipſius x. Sit porro Sy ſumma ſeu terminus ſummatorius ſeriei, exprimens aggregatum omnium terminorum a primo ſeu alio termino fixo uſque ad y incluſive. Computabimus autem ſummas ſerierum a termino primo, unde ſi ſit $x = 1$, dabit y terminum primum, atque Sy hunc y terminum primum exhibebit: ſin autem ponatur $x = 0$, terminus ſummatorius Sy in nihilum abire debet, propterea quod nulli termini ſummandi adſunt. Quocirca terminus ſummatorius Sy eiuſmodi erit functio ipſius x, quae evaneſcat poſito $x = 0$.

104. Si terminus generalis y ex pluribus partibus conſtet, ut ſit $y = p + q + r +$ &c. tum ipſa ſeries conſiderari poterit tanquam conflata ex pluribus aliis ſeriebus, quarum termini generales ſint p, q, r, &c. Hinc ſi ſingularum iſtarum ſerierum ſummae fuerint cognitae, ſimul ſeriei propoſitae ſumma poterit aſſignari; erit enim aggregatum ex ſummis ſingularum ſerierum. Hancobrem ſi ſit $y = p + q + r +$ &c. erit $Sy = Sp + Sq + Sr +$ &c. Cum igitur ſupra exhibuerimus ſummas ſerierum, quarum termini generales ſint quaecunque poteſtates ipſius x, habentes exponentes integros affirmativos, hinc cuiuſque ſeriei, cuius terminus generalis eſt $ax^{\alpha} + bx^{\beta} + cx^{\gamma} +$ &c. denotantibus α, β, γ, &c. numeros integros affirmativos, ſeu cuius terminus generalis eſt functio rationalis integra ipſius x, terminus ſummatorius inveniri poterit.

105. Sit in ſerie, cuius terminus generalis ſeu exponenti x reſpondens eſt $= y$, terminus hunc praecedens ſeu

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exponenti $x-1$ reſpondens $=v$, quoniam v oritur ex y, ſi loco x ſcribatur $x-1$; erit:

$$v=y-\frac{dy}{dx}+\frac{ddy}{2dx^2}-\frac{d^3y}{6dx^3}+\frac{d^4y}{24dx^4}-\frac{d^5y}{120dx^5}+\&c.$$

Si igitur y fuerit terminus generalis huius ſeriei

$$\overset{1}{a}+\overset{2}{b}+\overset{3}{c}+\overset{4}{d}+\ldots\ldots+\overset{x-1}{v}+\overset{x}{y}$$

huiusque ſeriei terminus indici o reſpondens fuerit $=A$, erit v, quatenus eſt functio ipſius x, terminus generalis huius ſeriei:

$$\overset{1}{A}+\overset{2}{a}+\overset{3}{b}+\overset{4}{c}+\overset{5}{d}+\ldots\ldots+\overset{x}{v}$$

unde ſi Sv denotet ſummam huius ſeriei, erit $Sv=Sy-y+A$. Sicque poſito $x=0$, quia fit $Sy=0$ & $y=A$, quoque Sv evaneſcet.

106. Cum igitur ſit $v=y-\frac{dy}{dx}+\frac{ddy}{2dx^2}-\frac{d^3y}{6dx^3}+\&c.$ erit per ante oſtenſa:

$$Sv=Sy-S\frac{dy}{dx}+S\frac{ddy}{2dx^2}-S\frac{d^3y}{6dx^3}+S\frac{d^4y}{24dx^4}-\&c.$$

atque ob $Sv=Sy-y+A$, erit:

$$y-A=S\frac{dy}{dx}-S\frac{ddy}{2dx^2}+S\frac{d^3y}{6dx^3}-S\frac{d^4y}{24dx^4}+\&c.$$

ideoque habebitur:

$$S\frac{dy}{dx}=y-A+S\frac{ddy}{2dx^2}-S\frac{d^3y}{6dx^3}+S\frac{d^4y}{24dx^4}-\&c.$$

Si ergo habeantur termini ſummatorii ſerierum, quarum termini generales ſunt $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, &c. ex iis obtinebitur terminus ſummatorius ſeriei, cuius terminus generalis eſt $\frac{dy}{dx}$. Quantitas vero conſtans A ita debet eſſe comparata, ut ſa-

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facto $x=0$ terminus ſummatorius $S\frac{dy}{dx}$ evaneſcat; hacque conditione facilius determinatur, quam ſi diceremus, eam eſſe terminum indici o reſpondentem in ſerie, cuius terminus generalis ſit $=y$.

107. Ex hoc fonte ſummae poteſtatum numerorum naturalium inveſtigari ſolent. Sit enim $y=x^{n+1}$; quoniam fit $\frac{dy}{dx}=(n+1)x^n$; $\frac{ddy}{2dx^2}=\frac{(n+1)n}{1.\ 2}x^{n-1}$;

$$\frac{d^3y}{6dx^3}=\frac{(n+1)n(n-1)}{1.\ 2.\ 3}x^{n-2};\ \frac{d^4y}{24dx^4}=\frac{(n+1)n(n-1)(n-2)}{1.\ 2.\ 3.\ 4}x^{n-3};\ \&c.$$

erit his valoribus ſubſtitutis:

$$(n+1)Sx^n=x^{n+1}-A+\frac{(n+1)n}{1.\ 2}Sx^{n-1}-\frac{(n+1)n(n-1)}{1.\ 2.\ 3.}Sx^{n-2}+\&c.$$

atque ſi utrinque per $n+1$ dividatur; erit:

$$Sx^n=\frac{1}{n+1}x^{n+1}+\frac{n}{2}Sx^{n-1}-\frac{n(n-1)}{2.\ 3}Sx^{n-2}+\frac{n(n-1)(n-2)}{2.\ 3.\ 4}Sx^{n-3}+\&c.$$
$$-\text{Conſt.}$$

quae conſtans ita accipi debet, ut poſito $x=0$, totus terminus ſummatorius evaneſcat. Ope huius ergo formulae ex iam cognitis ſummis poteſtatum inferiorum, quarum termini generales ſunt x^{n-1}, x^{n-2}, &c. inveniri poterit ſumma poteſtatum ſuperiorum termino generali x^n expreſſarum.

108. Si in hac expreſſione n denotet numerum integrum affirmativum, numerus terminorum erit finitus. Atque adeo hinc ſumma infinitarum poteſtatum ſi $n=0$, abſolute cognoſcetur; erit enim: $S.x^0=x$. Hacque cognita ad ſuperiores progredi licebit, poſito enim $n=1$; fiet:

$$S.x^1=\tfrac{1}{2}x^2+\tfrac{1}{2}Sx^0=\tfrac{1}{2}x^2+\tfrac{1}{2}x$$

ſi porro ponatur $n=2$ prodibit

$$S.x^2=\tfrac{1}{3}x+Sx-\tfrac{1}{3}Sx^0=\tfrac{1}{3}x^3+\tfrac{1}{2}x^2+\tfrac{1}{6}x;$$ deinde

$$S.x^3=\tfrac{1}{4}x^4+\tfrac{3}{2}Sx^2-Sx+\tfrac{1}{4}Sx^0=\tfrac{1}{4}x^4+\tfrac{1}{2}x^3+\tfrac{1}{4}x^2$$

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$S.x^4 = \frac{1}{5} x^5 + \frac{4}{2} Sx^3 - \frac{4}{2} Sx^2 + Sx - \frac{1}{5} Sx^0$ ſive
$S.x^4 = \frac{1}{5} x^5 + \frac{1}{2} x^4 + \frac{1}{3} x^3 - \frac{1}{30} x.$
Sicque porro quarumvis poteſtatum ſuperiorum ſummae ſucceſſivae ex inferioribus colligentur ; hoc autem facilius per ſequentes modos praeſtabitur.

109. Quoniam ſupra invenimus eſſe:

$$S\frac{dy}{dx} = y + \frac{1}{2} S\frac{ddy}{dx^2} - \frac{1}{6} S\frac{d^3y}{dx^3} + \frac{1}{24} S\frac{d^4y}{dx^4} - \frac{1}{120} S\frac{d^5y}{dx^5} + \&c.$$

Si ponamus $\frac{dy}{dx} = z$; fiet $\frac{ddy}{dx^2} = \frac{dz}{dx}$; $\frac{d^3y}{dx^3} = \frac{ddz}{dx^2}$; &c. tum vero ob $dy = zdx$, erit y quantitas, cuius differentiale eſt $= zdx$, quod hoc modo indicamus, ut ſit $y = \int zdx$. Quanquam autem haec inventio ipſius y ex dato z a calculo integrali pendet, tamen hic iam iſta forma $\int zdx$ uti poterimus, ſi quidem pro z alias ipſius x functiones non ſubſtituamus, niſi eiuſmodi, ut functio illa, cuius differentiale eſt $= zdx$, ex praecedentibus exhiberi queat. His igitur valoribus ſubſtitutis erit:

$$Sz = \int zdx + \frac{1}{2} S\frac{dz}{dx} - \frac{1}{6} S\frac{ddz}{dx^2} + \frac{1}{24} S\frac{d^3z}{dx^3} - \&c.$$

adiiciendo eiuſmodi conſtantem, ut poſito $x = 0$ ipſa ſumma Sz evaneſcat.

110. Subſtituendo autem loco y in ſuperiori expreſſione litteram z, vel quod eodem redit differentiando iſtam aequationem erit:

$$S\frac{dz}{dx} = z + \frac{1}{2} S\frac{ddz}{dx^2} - \frac{1}{6} S\frac{d^3z}{dx^3} + \frac{1}{24} S\frac{d^4z}{dx^4} - \&c.$$

ſin autem loco y ponatur $\frac{dz}{dx}$; erit:

$$S\frac{ddz}{dx^2} = \frac{dz}{dx} + \frac{1}{2} S\frac{d^3z}{dx^3} - \frac{1}{6} S\frac{d^4z}{dx^4} + \frac{1}{24} S\frac{d^5z}{dx^5} - \&c.$$

Similique modo ſi pro y ſucceſſive ponantur valores

$\frac{ddz}{dx^2}$; $\frac{d^3z}{dx^3}$; &c. reperietur:

$$S\frac{d^3z}{dx^3} = \frac{ddz}{dx^2} + \frac{1}{2}S\frac{d^4z}{dx^4} - \frac{1}{6}S\frac{d^5z}{dx^5} + \frac{1}{24}S\frac{d^6z}{dx^6} - \&c.$$

$$S\frac{d^4z}{dx^4} = \frac{d^3z}{dx^3} + \frac{1}{2}S\frac{d^5z}{dx^5} - \frac{1}{6}S\frac{d^6z}{dx^6} + \frac{1}{24}S\frac{d^7z}{dx^7} - \&c.$$

ſicque porro in infinitum.

111. Si nunc iſti valores pro $S\frac{dz}{dx}$; $S\frac{ddz}{dx^2}$; $S\frac{d^3z}{dx^3}$; &c. ſucceſſive ſubſtituantur in expreſſione:

$$Sz = \int z dx + \frac{1}{2}S\frac{dz}{dx} - \frac{1}{6}S\frac{ddz}{dx^2} + \frac{1}{24}S\frac{d^3z}{dx^3} - \&c.$$

invenietur expreſſio pro Sz, quae conſtabit ex his terminis $\int z dx$; z; $\frac{dz}{dx}$; $\frac{ddz}{dx^2}$; $\frac{d^3z}{dx^3}$; &c. quorum coefficientes facilius ſequenti modo inveſtigabuntur. Ponatur

$$Sz = \int z dz + \alpha z + \frac{\beta dz}{dx} + \frac{\gamma ddz}{dx^2} + \frac{\delta d^3z}{dx^3} + \frac{\varepsilon d^4z}{dx^4} + \&c.$$

atque pro his terminis ſui valores ſubſtituantur, quos obtinent ex praecedentibus ſeriebus, ex quibus eſt:

$$\int z dx = Sz - \frac{1}{2}S\frac{dz}{dx} + \frac{1}{6}S\frac{ddz}{dx^2} - \frac{1}{24}S\frac{d^3z}{dx^3} + \frac{1}{120}S\frac{d^4z}{dx^4} - \&c.$$

$$\alpha z = +\alpha S\frac{dz}{dx} - \frac{\alpha}{2}S\frac{ddz}{dx^2} + \frac{\alpha}{6}S\frac{d^3z}{dx^3} - \frac{\alpha}{24}S\frac{d^4z}{dx^4} + \&c.$$

$$\frac{\beta dz}{dx} = \beta S\frac{ddz}{dx^2} - \frac{\beta}{2}S\frac{d^3z}{dx^3} + \frac{\beta}{6}S\frac{d^4z}{dx^4} - \&c.$$

$$\frac{\gamma ddz}{dx^2} = \gamma S\frac{d^3z}{dx^3} - \frac{\gamma}{2}S\frac{d^4z}{dx^4} + \&c.$$

$$\frac{\delta d^3z}{dx^3} = \delta S\frac{d^4z}{dx^4} - \&c.$$

qui

qui valores additi, cum producere debeant Sz, coefficientes α, β, γ, δ, &c. ex ſequentibus aequationibus definientur:

$$\alpha - \frac{1}{2} = 0$$

$$\beta - \frac{\alpha}{2} + \frac{1}{6} = 0$$

$$\gamma - \frac{\beta}{2} + \frac{\alpha}{6} - \frac{1}{24} = 0$$

$$\delta - \frac{\gamma}{2} + \frac{\beta}{6} - \frac{\alpha}{24} + \frac{1}{120} = 0$$

$$\varepsilon - \frac{\delta}{2} + \frac{\gamma}{6} - \frac{\beta}{24} + \frac{\alpha}{120} - \frac{1}{720} = 0$$

$$\zeta - \frac{\varepsilon}{2} + \frac{\delta}{6} - \frac{\gamma}{24} + \frac{\beta}{120} - \frac{\alpha}{720} + \frac{1}{5040} = 0$$

112. Ex his ergo aequationibus ſucceſſive valores omnium litterarum α, β, γ, δ, &c. definiri poterunt, reperietur autem:

$$\alpha = \frac{1}{2}$$

$$\beta = \frac{\alpha}{2} - \frac{1}{6} = \frac{1}{12}$$

$$\gamma = \frac{\beta}{2} - \frac{\alpha}{6} + \frac{1}{24} = 0$$

$$\delta = \frac{\gamma}{2} - \frac{\beta}{6} + \frac{\alpha}{24} - \frac{1}{120} = -\frac{1}{720}$$

$$\varepsilon = \frac{\delta}{2} - \frac{\gamma}{6} + \frac{\beta}{24} - \frac{\alpha}{120} + \frac{1}{720} = 0$$

&c.

ſicque ulterius progrediendo reperientur continuo termini alterni evaneſcentes. Litterae ergo ordine tertia, quinta, ſeptima,

ma, &c. omneſque impares erunt $=0$, excepta prima, quo ipſo haec valorum ſeries contra legem continuitatis impingere videtur. Quamobrem eo magis neceſſe eſt, ut rigide demonſtretur, omnes terminos impares praeter primum neceſſario evaneſcere.

113. Quoniam ſingulae litterae ſecundum legem conſtantem ex praecedentibus determinantur, eae ſeriem recurrentem inter ſe conſtituent. Ad quam explicandam concipiatur iſta ſeries: $1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \&c.$ cuius valor ſit $= V$; atque manifeſtum eſt hanc ſeriem recurrentem oriri ex evolutione huius fractionis:

$$V = \frac{1}{1 - \frac{1}{2}u + \frac{1}{6}u^2 - \frac{1}{24}u^3 + \frac{1}{120}u^4 - \&c.}$$

atque ſi iſta fractio alio modo in ſeriem infinitam ſecundum poteſtates ipſius u progredientem reſolvi queat, neceſſe eſt, ut ſemper eadem ſeries:

$$V = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \&c.$$

reſultet: hocque modo alia lex, qua iſti iidem valores α, β, γ, δ, &c. determinantur, eruetur.

114. Quoniam, ſi e denotet numerum, cuius logarithmus hyperbolicus unitati aequatur, erit:

$$e^{-u} = 1 - u + \frac{1}{2}u^2 - \frac{1}{6}u^3 + \frac{1}{24}u^4 - \frac{1}{120}u^5 + \&c.$$

erit: $\dfrac{1 - e^{-u}}{u} = 1 - \frac{1}{2}u + \frac{1}{6}u^2 - \frac{1}{24}u^3 + \frac{1}{120}u^4 - \&c.$

ideoque $V = \dfrac{u}{1 - e^{-u}}$. Nunc extinguatur ex ſerie ſecundus terminus $\alpha u = \frac{1}{2}u$, ut ſit:

$V - \frac{1}{2}u = 1 + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \&c.$ erit:

$V - \frac{1}{2}u = \dfrac{\frac{1}{2}u(1 + e^{-u})}{1 - e^{-u}}$. Multiplicentur numerator ac denominator per $e^{\frac{1}{2}u}$, eritque $V - \frac{1}{2}u = \dfrac{u(e^{\frac{1}{2}u} + e^{-\frac{1}{2}u})}{2(e^{\frac{1}{2}u} - e^{-\frac{1}{2}u})}$,

& quantitatibus $e^{\frac{1}{2}u}$ & $e^{-\frac{1}{2}u}$ in ſeries converſis fiet

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$$V - \tfrac{1}{2}u = \frac{1 + \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} + \frac{u^6}{2.4.6.8.10.12} + \&c.}{2(\frac{1}{2} + \frac{u^2}{2.4.6} + \frac{u^4}{2.4.6.8.10} + \&c.)}$$

ſive

$$V - \tfrac{1}{2}u = \frac{1 + \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} + \frac{u^6}{2.4\ldots12} + \frac{u^8}{2.4\ldots16} + \&c.}{1 + \frac{u^2}{4.6} + \frac{u^4}{4.6.8.10} + \frac{u^6}{4.6\ldots14} + \frac{u^8}{4.6\ldots18} + \&c.}$$

115. Cum igitur in hac fractione poteſtates impares prorſus deſint, in eius quoque evolutione poteſtates impares omnino nullae ingredientur; quare cum $V - \frac{1}{2}u$ aequetur iſti ſeriei:

$$1 + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \&c.$$

coefficientes imparium poteſtatum γ, ε, η, ι, &c. omnes evaneſcent. Sicque ratio manifeſta eſt, cur in ſerie:

$$1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \&c.$$

termini ordine pares omnes praeter ſecundum ſint $=0$, neque tamen lex continuitatis vim patiatur. Erit ergo

$$V = 1 + \tfrac{1}{2}u + \beta u^2 + \delta u^4 + \zeta u^6 + \theta u^8 + \varkappa u^{10} + \&c.$$

litteriſque β, δ, ζ, θ, $\varkappa$, &c. per evolutionem ſuperioris fractionis determinatis, obtinebimus terminum ſummatorium Sz ſeriei, cuius terminus generalis eſt $=z$, indici x reſpondens, hoc modo expreſſum:

$$Sz = \int z dx + \tfrac{1}{2}z + \frac{\beta dz}{dx} + \frac{\delta d^3 z}{dx^3} + \frac{\zeta d^5 z}{dx^5} + \frac{\theta d^7 z}{dx^7} + \&c.$$

116. Quia ſeries $1 + \beta u^2 + \delta u^4 + \zeta u^6 + \theta u^8 + \&c.$ oritur ex evolutione huius fractionis:

$$\frac{1 + \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} + \frac{u^6}{2.4.6.8.10.12} + \&c.}{1 + \frac{u^2}{4.6} + \frac{u^4}{4.6.8.10} + \frac{u^6}{4.6.8.10.12.14} + \&c.}$$

lit-

litterae β, δ, ζ, θ, &c. hanc legem tenebunt, ut ſit:

$$\beta = \frac{1}{2.4} - \frac{1}{4.6}$$

$$\delta = \frac{1}{2.4.6.8} - \frac{\beta}{4.6} - \frac{1}{4.6.8.10}$$

$$\zeta = \frac{1}{2.4.6\ldots12} - \frac{\delta}{4.6} - \frac{\beta}{4.6.8.10} - \frac{1}{4.6\ldots14}$$

$$\theta = \frac{1}{2.4\ldots16} - \frac{\zeta}{4.6} - \frac{\delta}{4.6.8.10} - \frac{\beta}{4.6\ldots14} - \frac{1}{4.6\ldots18}.$$

Hi autem valores alternative fiunt affirmativi & negativi.

117. Si igitur harum litterarum alternae capiantur negative, ita ut ſit:

$$Sz = \int z dx + \tfrac{1}{2} z - \frac{\beta dz}{dx} + \frac{\delta d^3 z}{dx^3} - \frac{\zeta d^5 z}{dx^5} + \frac{\theta d^7 z}{dx^7} - \&c.$$

litterae β, δ, ζ, θ, &c. definientur ex hac fractione:

$$\frac{1 - \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} - \frac{u^6}{2.4\ldots12} + \frac{u^8}{2.4\ldots16} - \&c.}{1 - \frac{u^2}{4.6} + \frac{u^4}{4.6.8.10} - \frac{u^6}{4.6\ldots14} + \frac{u^8}{4.6\ldots18} - \&c.}$$

eam evolvendo in ſeriem:

$$1 + \beta u^2 + \delta u^4 + \zeta u^6 + \theta u^8 + \&c.$$ quocirca erit;

$$\beta = \frac{1}{4.6} - \frac{1}{2.4}$$

$$\delta = \frac{\beta}{4.6} - \frac{1}{4.6.8.10} + \frac{1}{2.4.6.8}$$

$$\zeta = \frac{\delta}{4.6} - \frac{\beta}{4.6.8.10} + \frac{1}{4.6\ldots14} - \frac{1}{2.4\ldots12}$$

&c.

nunc autem omnes termini fient negativi.

118. Ponamus ergo $\beta = -A$; $\delta = -B$; $\zeta = -C$; &c. ut ſit:

$$Sz = \int z dx + \tfrac{1}{2} z + \frac{A dz}{dx} - \frac{B d^3 z}{dx^3} + \frac{C d^5 z}{dx^5} - \frac{D d^7 z}{dx^7} + \&c.$$

atque ad litteras A, B, C, D, &c. definiendas conſideretur haec ſeries:

$$1 - Au^2 - Bu^4 - Cu^6 - Du^8 - Eu^{10} - \&c.$$

quae oritur ex evolutione huius fractionis:

$$\frac{1 - \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} - \frac{u^6}{2.4 \ldots 12} + \frac{u^8}{2.4 \ldots 16} - \&c.}{1 - \frac{u^2}{4.6} + \frac{u^4}{4.6.8.10} - \frac{u^6}{4.6 \ldots 14} + \frac{u^8}{4.6 \ldots 18} - \&c.}$$

vel conſideretur iſta ſeries:

$$\frac{1}{u} - Au - Bu^3 - Cu^5 - Du^7 - Eu^9 - \&c. = s$$

quae oritur ex evolutione huius fractionis:

$$s = \frac{1 - \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} - \frac{u^6}{2.4 \ldots 12} + \&c.}{u - \frac{u^3}{4.6} + \frac{u^5}{4.6.8.10} - \frac{u^7}{4.6 \ldots 14} + \&c.}$$

Cum autem ſit:

$$\operatorname{coſ} \tfrac{1}{2} u = 1 - \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} - \frac{u^6}{2.4 \ldots 12} + \&c.$$

$$\operatorname{ſin} \tfrac{1}{2} u = \frac{u}{2} - \frac{u^3}{2.4.6} + \frac{u^5}{2.4.6.8.10} - \frac{u^7}{2.4 \ldots 14} + \&c.$$

ſequitur fore: $s = \dfrac{\operatorname{coſ} \frac{1}{2} u}{2 \operatorname{ſin} \frac{1}{2} u} = \frac{1}{2} \cot. \frac{1}{2} u.$

Quare ſi cotangens arcus $\frac{1}{2} u$ in ſeriem convertatur, cuius termini ſecundum poteſtates ipſius u procedant, ex ea cognoſcentur valores litterarum A, B, C, D, E, &c.

119. Cum igitur ſit $s = \frac{1}{2} \cot \frac{1}{2} u$; erit $\frac{1}{2} u =$ A cot $2s$, & differentiando erit $\frac{1}{2} du = \dfrac{-2 ds}{1 + 4ss}$ ſeu

4ds

$4ds + du + 4ssdu = 0$, ſive $\frac{4ds}{du} + 1 + 4ss = 0$.

Quia autem eſt: $s = \frac{1}{u} - Au - Bu^3 - Cu^5 - \&c.$

erit

$$\begin{array}{rl}
\frac{4ds}{du} = & -\frac{4}{uu} - 4A - 3.4Bu^2 - 5.4Cu^4 - 7.4Du^6 - \&c. \\
1 = & 1 \\
4ss = & \frac{4}{uu} - 8A - 8Bu^2 - 8Cu^4 - 8Du^6 - \&c. \\
 & \qquad + 4A^2u^2 + 8ABu^4 + 8ACu^6 + \&c. \\
 & \qquad\qquad + 4BBu^6 + \&c.
\end{array}$$

perductis his terminis homogeneis ad cyphram fiet:

$$A = \frac{1}{12}$$

$$B = \frac{A^2}{5}$$

$$C = \frac{2AB}{7}$$

$$D = \frac{2AC + BB}{9}$$

$$E = \frac{2AD + 2BC}{11}$$

$$F = \frac{2AE + 2BD + CC}{13}$$

$$G = \frac{2AF + 2BE + 2CD}{15}$$

$$H = \frac{2AG + 2BF + 2CE + DD}{17}$$

&c.

Ex quibus formulis iam manifeſto liquet, ſingulos hos valores eſſe affirmativos.

120. Quoniam vero denominatores horum valorum fiunt vehementer magni, calculumque non mediocriter impediunt; loco litterarum A, B, C, D, &c. has novas introducamus:

$$A = \frac{\alpha}{1.2.3}$$

$$B = \frac{\beta}{1.2.3.4.5}$$

$$C = \frac{\gamma}{1.2.3\ldots.7}$$

$$D = \frac{\delta}{1.2.3\ldots..9}$$

$$E = \frac{\varepsilon}{1.2.3\ldots\ldots 11} \qquad \&c.$$

Atque reperietur fore:

$$\alpha = \frac{1}{2}$$

$$\beta = \frac{2}{3}\alpha^2$$

$$\gamma = 2.\frac{3}{3}\alpha\beta$$

$$\delta = 2.\frac{4}{3}\alpha\gamma + \frac{8.7}{4.5}\beta^2$$

$$\varepsilon = 2.\frac{5}{3}\alpha\delta + 2.\frac{10.9.8}{1\ldots.5}\beta\gamma$$

$$\zeta = 2.\frac{12}{1.2.3.}\alpha\varepsilon + 2.\frac{12.11.10}{1\ldots..5}\beta\delta + \frac{12.11.10.9.8}{1\ldots\ldots.7}\gamma\gamma$$

$$\eta = 2.\frac{14}{1.2.3}\alpha\zeta + 2.\frac{14.13.12}{1\ldots..5}\beta\varepsilon + 2.\frac{14.13.12.11.10}{1\ldots\ldots\ldots 7}\gamma\delta$$

&c.

121. Commodius autem utemur his formulis:

$$\alpha = \frac{1}{2}$$

$$\beta = \frac{4}{3}\cdot\frac{\alpha\alpha}{2}$$

$$\gamma = \frac{6}{3}\cdot\alpha\beta$$

$$\delta = \frac{8}{3}\cdot\alpha\gamma + \frac{8.\ 7.\ 6}{3.\ 4.\ 5}\cdot\frac{\beta\beta}{2}$$

$$\varepsilon = \frac{10}{3}\cdot\alpha\delta + \frac{10.9.\ 8}{3.\ 4.\ 5}\cdot\beta\gamma$$

$$\zeta = \frac{12}{3}\cdot\alpha\varepsilon + \frac{12.11.10}{3.\ 4.\ 5}\cdot\beta\delta + \frac{12.11.10.9.8}{3.\ 4.\ 5.\ 6.\ 7}\cdot\frac{\gamma\gamma}{2}$$

$$\eta = \frac{14}{3}\cdot\alpha\zeta + \frac{14.13.12}{3.\ 4.\ 5}\cdot\beta\varepsilon + \frac{14.13\ldots10}{3.4\ldots7}\cdot\gamma\delta$$

$$\theta = \frac{16}{3}\cdot\alpha\eta + \frac{16.15.14}{3.\ 4.\ 5}\cdot\beta\zeta + \frac{16.15\ldots12}{3.4\ldots.7}\cdot\gamma\varepsilon + \frac{16.15\ldots10}{3.\ 4\ldots9}\cdot\frac{\delta\delta}{2}$$

&c.

Ex hac igitur lege, ſecundum quam calculus non difficulter inſtituitur, ſi inventi fuerint valores litterarum α, β, γ, δ, &c. tum ſeriei cuiuſcunque; cuius terminus generalis ſeu indici x conveniens fuerit $= z$, terminus ſummatorius ita exprimetur, ut ſit:

$$Sz = \int z dx + \tfrac{1}{2}z + \frac{\alpha dz}{1.2.3\,dx} - \frac{\beta d^3 z}{1.2.3.4.5\,dx^3} + \frac{\gamma d^5 z}{1.2\ldots.7\,dx^5}$$

$$- \frac{\delta d^7 z}{1.2\ldots\ldots9dx^7} + \frac{\varepsilon d^9 z}{1.2\ldots.11dx^9} - \frac{\zeta d^{11} z}{1.2\ldots.13dx^{11}} + \&c.$$

iſtae autem litterae α, β, γ, δ, &c. ſequentes valores habere inventae ſunt:

$\alpha =$

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$$\alpha = \frac{1}{2} \qquad 1.2\,\alpha = 1$$

$$\beta = \frac{1}{6} \qquad 1.2.3\,\beta = 1$$

$$\gamma = \frac{1}{6} \qquad 1.2.3.4\,\gamma = 4$$

$$\delta = \frac{3}{10} \qquad 1.2.3.4.5\,\delta = 36$$

$$\varepsilon = \frac{5}{6} \qquad 1.2.3\ldots 6\,\varepsilon = 600$$

$$\zeta = \frac{691}{210} \qquad 1.2.3\ldots 7\,\zeta = 24.691$$

$$\eta = \frac{35}{2} \qquad 1.2.3\ldots 8\,\eta = 20160.35$$

$$\theta = \frac{3617}{30} \qquad 1.2.3\ldots 9\,\theta = 12096.3617$$

$$\iota = \frac{43867}{42} \qquad 1.2.3\ldots 10\,\iota = 86400.43867$$

$$\varkappa = \frac{1222277}{110} \qquad 1.2.3\ldots 11\,\varkappa = 362880.1222277$$

$$\lambda = \frac{854513}{6} \qquad 1.2.3\ldots 12\,\lambda = 79833600.854513$$

$$\mu = \frac{1181820455}{546} \qquad 1.2.3\ldots 13\,\mu = 11404800.1181820455$$

$$\nu = \frac{76977927}{2} \qquad 1.2.3\ldots 14\,\nu = 43589145600.76977927$$

$$\xi = \frac{23749461029}{30} \qquad 1.2.3\ldots 15\,\xi = 43589145600.23749461029$$

$$\pi = \frac{8615841276005}{462} \qquad 1.2.3\ldots 16\,\pi = 45287424000.8615841276005$$

&c.

122. Numeri iſti per univerſam ſerierum doctrinam ampliſſimum habent uſum. Primum enim ex his numeris formari poſſunt ultimi termini in ſummis poteſtatum parium, quos non aeque ac reliquos terminos ex ſummis praecedentium reperiri poſſe ſupra annotavimus. In poteſtatibus enim paribus poſtremi ſummarum termini ſunt x per certos numeros multiplicati; qui numeri pro poteſtatibus II; IV; VI; VIII; &c. ſunt $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, $\frac{1}{30}$, &c. ſignis alternantibus affecti. Oriuntur autem hi numeri ſi valores litterarum α, β, γ, δ, &c. ſupra inventi reſpective dividantur per numeros impares 3, 5, 7, &c. unde iſti numeri, qui ab Inventore *Iacobo Bernoullio* vocari ſolent Bernoulliani erunt:

$$\frac{\alpha}{3} = \frac{1}{6} = \mathfrak{A}$$

$$\frac{\beta}{5} = \frac{1}{30} = \mathfrak{B}$$

$$\frac{\gamma}{7} = \frac{1}{42} = \mathfrak{C}$$

$$\frac{\delta}{9} = \frac{1}{30} = \mathfrak{D}$$

$$\frac{\varepsilon}{11} = \frac{5}{66} = \mathfrak{E}$$

$$\frac{\zeta}{13} = \frac{691}{2730} = \mathfrak{F}$$

$$\frac{\eta}{15} = \frac{7}{6} = \mathfrak{G}$$

$$\frac{\theta}{17} = \frac{3617}{510} = \mathfrak{H}$$

$$\frac{\iota}{19} = \frac{43867}{798} = \mathfrak{I}$$

$$\frac{\varkappa}{21} = \frac{174611}{330} = \mathfrak{K} = \frac{283.617}{330}$$

$$\frac{\lambda}{23} = \frac{854513}{138} = \mathfrak{L} = \frac{11.131.593}{2.3.23}$$

$$\frac{\mu}{25} = \frac{236364091}{2730} = \mathfrak{M}$$

$$\frac{\nu}{27} = \frac{8553103}{6} = \mathfrak{N} = \frac{13.657931}{6}$$

$$\frac{\xi}{29} = \frac{23749461029}{870} = \mathfrak{O}$$

$$\frac{\pi}{31} = \frac{8615841276005}{14322} = \mathfrak{P}$$

&c.

123. Iſti igitur numeri Bernoulliani $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. immediate ex ſequentibus aequationibus inveniri poterunt:

$$\mathfrak{A} = \frac{1}{6}$$

$$\mathfrak{B} = \frac{4.3}{1.2} \cdot \frac{1}{5} \mathfrak{A}^2$$

$$\mathfrak{C} = \frac{6.5}{1.2} \cdot \frac{2}{7} \mathfrak{A}\mathfrak{B}$$

$$\mathfrak{D} = \frac{8.7}{1.2} \cdot \frac{2}{9} \mathfrak{A}\mathfrak{C} + \frac{8.7.6.5}{1.2.3.4} \cdot \frac{1}{9} \mathfrak{B}^2$$

$$\mathfrak{E} = \frac{10.9}{1.2} \cdot \frac{2}{11} \mathfrak{A}\mathfrak{D} + \frac{10.9.8.7}{1.2.3.4} \cdot \frac{2}{11} \mathfrak{B}\mathfrak{C}$$

$$\mathfrak{F} = \frac{12.11}{1.2} \cdot \frac{2}{13} \mathfrak{A}\mathfrak{E} + \frac{12.11.10.9}{1.2.3.4} \cdot \frac{2}{13} \mathfrak{B}\mathfrak{D} + \frac{12.11.10.9.8.7}{1.2.3.4.5.6} \cdot \frac{1}{13} \mathfrak{C}^2$$

$$\mathfrak{G} = \frac{14.13}{1.2} \cdot \frac{2}{15} \mathfrak{A}\mathfrak{F} + \frac{14.13.12.11}{1.2.3.4} \cdot \frac{2}{15} \mathfrak{B}\mathfrak{E} + \frac{14.13.12.11.10.9}{1.2.3.4.5.6} \cdot \frac{2}{15} \mathfrak{C}\mathfrak{D}$$

&c.

qua-

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quarum aequationum lex per ſe eſt manifeſta, ſi tantum notetur, ubi quadratum cuiuſpiam litterae occurrit, eius coefficientem duplo eſſe minorem, quam ſecundum regulam eſſe debere videatur. Revera autem termini, qui continent producta ex diſparibus litteris, bis occurrere cenſendi ſunt, erit enim verbi gratia:

$$13\mathfrak{J} = \frac{12.11}{1.2}\mathfrak{A}\mathfrak{E} + \frac{12.11.10.9}{1.2.3.4}\mathfrak{B}\mathfrak{D} + \frac{12.11.10.9.8.7}{1.2.3.4.5.6}\mathfrak{C}\mathfrak{C} + \frac{12.11.10 \ldots 5}{1.2.3 \ldots 8}\mathfrak{D}\mathfrak{B} + \frac{12.11.10 \ldots 3}{1.2.3 \ldots 10}\mathfrak{E}\mathfrak{A}$$

124. Deinde vero etiam iidem numeri α, β, γ, δ, &c. ingrediuntur in expreſſiones ſummarum ſerierum fractionum in hac forma generali:

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \&c.$$

quoties n eſt numerus par affirmativus; contentarum. Has enim ſummas in introductione per poteſtates ſemiperipheriae circuli π radio exiſtente $= 1$ expreſſas dedimus, atque in harum poteſtatum coefficientibus iſti ipſi numeri $\alpha, \beta, \gamma, \delta$, &c. ingredi deprehenduntur. Quo autem haec convenientia non caſu evenire, ſed neceſſario locum habere intelligatur, has eaſdem ſummas ſingulari modo inveſtigemus, quo lex ſummarum illarum facilius patebit. Quoniam ſupra invenimus eſſe:

$$\frac{\pi}{n}\cot.\frac{m}{n}\pi = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \&c.$$

binis terminis coniungendis habebimus:

$$\frac{\pi}{n}\cot.\frac{m}{n}\pi = \frac{1}{m} - \frac{2m}{nn-m^2} - \frac{2m}{4n^2-m^2} - \frac{2m}{9n^2-m^2} - \frac{2m}{16n^2-m^2} - \&c.$$

unde colligimus fore:

$$\frac{1}{n^2-m^2} + \frac{1}{4n^2-m^2} + \frac{1}{9n^2-m^2} + \frac{1}{16n^2-m^2} + \&c. = \frac{1}{2mm} - \frac{\pi}{2mn}\cot.\frac{m}{n}\pi$$

 Sta-

Statuamus nunc $n = 1$, & pro m ponamus u; ut ſit:

$$\frac{1}{1-u^2} + \frac{1}{4-u^2} + \frac{1}{9-u^2} + \frac{1}{16-u^2} + \&c. = \frac{1}{2uu} - \frac{\pi}{2u}\cot.\pi u.$$

Reſolvantur ſingulae iſtae fractiones in ſeries:

$$\frac{1}{1-u^2} = 1 + u^2 + u^4 + u^6 + u^8 + \&c.$$

$$\frac{1}{4-u^2} = \frac{1}{2^2} + \frac{u^2}{2^4} + \frac{u^4}{2^6} + \frac{u^6}{2^8} + \frac{u^8}{2^{10}} + \&c.$$

$$\frac{1}{9-u^2} = \frac{1}{3^2} + \frac{u^2}{3^4} + \frac{u^4}{3^6} + \frac{u^6}{3^8} + \frac{u^8}{3^{10}} + \&c.$$

$$\frac{1}{16-u^2} = \frac{1}{4^2} + \frac{u^2}{4^4} + \frac{u^4}{4^6} + \frac{u^6}{4^8} + \frac{u^8}{4^{10}} + \&c.$$

&c.

125. Quod ſi ergo ponatur:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c. = \mathfrak{a}$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \&c. = \mathfrak{b}$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \&c. = \mathfrak{c}$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \&c. = \mathfrak{d}$$

$$1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \&c. = \mathfrak{e}$$

&c.

ſuperior ſeries tranſmutabitur in hanc:

$$\mathfrak{a} + \mathfrak{b}u^2 + \mathfrak{c}u^4 + \mathfrak{d}u^6 + \mathfrak{e}u^8 + \mathfrak{f}u^{10} + \&c. = \frac{1}{2uu} - \frac{\pi}{2u}\cot.\pi u.$$

Cum igitur in §. 118. litterae A, B, C, D, &c. ita comparatae ſint inventae, ut poſito:

$s =$

$$s = \frac{1}{u} - Au - Bu^3 - Cu^5 - Du^7 - Eu^9 - \&c.$$

ſit $s = \frac{1}{2}\cot.\frac{1}{2}u$, erit poſito πu loco $\frac{1}{2}u$ ſeu $2\pi u$ loco u

$$\tfrac{1}{2}\cot\pi u = \frac{1}{2\pi u} - 2A\pi u - 2^3B\pi^3u^3 - 2^5C\pi^5u^5 - 2^7D\pi^7u^7 - \&c.$$

unde per $\frac{\pi}{u}$ multiplicando erit:

$$\frac{\pi}{2u}\cot.\pi u = \frac{1}{2uu} - 2A\pi^2 - 2^3B\pi^4u^2 - 2^5C\pi^6u^4 - \&c.$$

hincque ſequitur fore:

$$\frac{1}{2uu} - \frac{\pi}{2u}\cot.\pi u = 2A\pi^2 + 2^3B\pi^4u^2 + 2^5C\pi^6u^4 + 2^7D\pi^8u^6 + \&c.$$

Quia igitur modo invenimus eſſe:

$$\frac{1}{2uu} - \frac{\pi}{2u}\cot.\pi u = \mathfrak{a} + \mathfrak{b}u^2 + \mathfrak{c}u^4 + \mathfrak{d}u^6 + \&c.$$

neceſſe eſt ut ſit:

$$\mathfrak{a} = 2\,A\,\pi^2 = \frac{2\alpha}{1.2.3}.\pi^2 = \frac{2\mathfrak{A}}{1.2}.\pi^2$$

$$\mathfrak{b} = 2^3\,B\,\pi^4 = \frac{2^3\beta}{1.2.3.4.5}.\pi^4 = \frac{2^3\mathfrak{B}}{1.2.3.4}.\pi^4$$

$$\mathfrak{c} = 2^5\,C\,\pi^6 = \frac{2^5\gamma}{1.2.3\ldots7}\pi^6 = \frac{2^5\mathfrak{C}}{1.2\ldots\ldots6}\pi^6$$

$$\mathfrak{d} = 2^7\,D\,\pi^8 = \frac{2^7\delta}{1.2.3\ldots9}\pi^8 = \frac{2^7\mathfrak{D}}{1.2\ldots\ldots8}\pi^8$$

$$\mathfrak{e} = 2^9\,E\,\pi^{10} = \frac{2^9\varepsilon}{1.2.3\ldots11}\pi^{10} = \frac{2^9\mathfrak{E}}{1.2\ldots.10}\pi^{10}$$

$$\mathfrak{f} = 2^{11}F\,\pi^{12} = \frac{2^{11}\zeta}{1.2.3\ldots13}\pi^{12} = \frac{2^{11}\mathfrak{F}}{1.2\ldots.12}\pi^{12}\quad\&c.$$

126. Ex hoc ergo tam facili ratiocinio non ſolum omnes ſeries poteſtatum reciprocarum, quas §. praeced. exhi-

buimus, expedite ſummantur; ſed ſimul quoque perſpicitur, quemadmodum iſtae ſummae ex cognitis valoribus litterarum α, β, γ, δ, ε, &c. vel etiam ex numeris Bernoullianis $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. formentur. Quare cum iſtorum numerorum quindecim §. 122. definiverimus, ex iis ſummae omnium poteſtatum parium uſque ad ſummam huius ſeriei incluſive aſſignari poterunt:

$1 + \frac{1}{2^{30}} + \frac{1}{3^{30}} + \frac{1}{4^{30}} + \frac{1}{5^{30}} +$ &c. erit enim huius ſeriei ſumma $= \frac{2^{29}\pi}{1.2.3\ldots31}\,\pi^{30} = \frac{2^{29}\mathfrak{P}}{1.2\ldots\ldots30}\,\pi^{30}$.

Atque ſi quis voluerit has ſummas ulterius determinare, id continuandis numeris α, β, γ, &c. vel his $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. facillime praeſtabitur.

127. Origo ergo horum numerorum α, β, γ, δ, &c. vel inde formatorum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. potiſſimum debetur evolutioni cotangentis cuiuſvis anguli in ſeriem infinitam. Cum enim ſit

$$\tfrac{1}{2}\cot.\tfrac{1}{2}u = \frac{1}{u} - Au - Bu^3 - Cu^5 - Du^7 - Eu^9 - \&c.$$

erit:

$$Au^2 + Bu^4 + Cu^6 + Du^8 + \&c. = 1 - \frac{u}{2}\cot.\tfrac{1}{2}u,$$

ſi igitur loco coefficientium A, B, C, D, &c. valores ipſorum ſubſtituantur, reperietur:

$$\frac{\alpha u^2}{1.2.3} + \frac{\beta u^4}{1.2\ldots5} + \frac{\gamma u^6}{1.2\ldots.7} + \frac{\delta u^8}{1.2\ldots9} + \&c. = 1 - \frac{u}{2}\cot\tfrac{1}{2}u$$

atque numeros Bernoullianos adhibendo erit:

$$\frac{\mathfrak{A}u^2}{1.2} + \frac{\mathfrak{B}u^4}{1.2.3.4} + \frac{\mathfrak{C}u^6}{1.2\ldots6} + \frac{\mathfrak{D}u^8}{1.2\ldots.8} + \&c. = 1 - \frac{u}{2}\cot\tfrac{1}{2}u$$

ex quibus ſeriebus per differentiationem innumerabiles aliae deduci poſſunt, ſicque infinitae ſeries ſummari, in quas iſti numeri notatu tantopere digni ingrediuntur.

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128. Sumamus aequationem priorem, quam per u multiplicemus, ut ſit:

$$\frac{\alpha u^3}{1.2.3} + \frac{\beta u^5}{1.2\ldots5} + \frac{\gamma u^7}{1.2\ldots7} + \frac{\delta u^9}{1.2\ldots9} + \&c. = u - \frac{uu}{2}\cot\tfrac{1}{2}u$$

quae differentiata ac per du diviſa dat:

$$\frac{\alpha u^2}{1.2} + \frac{\beta u^4}{1.2.3.4} + \frac{\gamma u^6}{1.2\ldots6} + \frac{\delta u^8}{1.2\ldots8} + \&c. = 1 - u\cot\tfrac{1}{2}u + \frac{uu}{4(\sin\frac{1}{2}u)^2}$$

&, ſi denuo differentietur erit:

$$\frac{\alpha u}{1} + \frac{\beta u^3}{1.2.3} + \frac{\gamma u^5}{1.2.3.4.5} + \&c. = -\cot\tfrac{1}{2}u + \frac{u}{(\sin\frac{1}{2}u)^2} - \frac{uu\cos\frac{1}{2}u}{4(\sin\frac{1}{2}u)^3}$$

Sin autem altera aequatio differentietur erit:

$$\frac{\mathfrak{A}u}{1} + \frac{\mathfrak{B}u^3}{1.2.3} + \frac{\mathfrak{C}u^5}{1.2\ldots5} + \frac{\mathfrak{D}u^7}{1.2\ldots7} = -\tfrac{1}{2}\cot\tfrac{1}{2}u + \frac{u}{4(\sin\frac{1}{2}u)^2}$$

Ex his ergo ſi ponatur $u = \pi$, ob $\cot\frac{1}{2}\pi = 0$, & $\sin\frac{1}{2}u = 1$, ſequuntur iſtae ſummationes:

$$1 = \frac{\alpha\pi^2}{1.2.3} + \frac{\beta\pi^4}{1.2.3.4.5} + \frac{\gamma\pi^6}{1.2.3\ldots7} + \frac{\delta\pi^8}{1.2.3\ldots9} + \&c.$$

$$1 + \frac{\pi^2}{4} = \frac{\alpha\pi^2}{1.2} + \frac{\beta\pi^4}{1.2.3.4} + \frac{\gamma\pi^6}{1.2.3\ldots6} + \frac{\delta\pi^8}{1.2.3\ldots8} + \&c.$$

$$\pi = \frac{\alpha\pi}{1} + \frac{\beta\pi^3}{1.2.3} + \frac{\gamma\pi^5}{1.2.3.4.5} + \frac{\delta\pi^7}{1.2.3\ldots7} + \&c.$$

ſeu $$1 = \alpha + \frac{\beta\pi^2}{1.2.3} + \frac{\gamma\pi^4}{1.2.3.4.5} + \frac{\delta\pi^6}{1.2.3\ldots7} + \&c.$$

a qua ſi prima ſubtrahatur remanebit:

$$\alpha = \frac{(\alpha-\beta)\pi^2}{1.2.3} + \frac{(\beta-\gamma)\pi^4}{1.2.3.4.5} + \frac{(\gamma-\delta)\pi^6}{1.2.3\ldots7} + \&c.$$

Tum vero erit:

$$1 = \frac{\mathfrak{A}\pi^2}{1.2} + \frac{\mathfrak{B}\pi^4}{1.2.3.4} + \frac{\mathfrak{C}\pi^6}{1.2.3\ldots6} + \frac{\mathfrak{D}\pi^8}{1.2.3\ldots8} + \&c.$$

$$\frac{\pi}{4} = \frac{\mathfrak{A}\pi}{1} + \frac{\mathfrak{B}\pi^3}{1.2.3} + \frac{\mathfrak{C}\pi^5}{1.2.3.4.5} + \frac{\mathfrak{D}\pi^7}{1.2.3\ldots7} + \&c.$$

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ſeu $\frac{1}{4} = \frac{\mathfrak{A}}{1} + \frac{\mathfrak{B}\pi^2}{1.\,2.\,3} + \frac{\mathfrak{C}\pi^4}{1.2.3.4.5} + \frac{\mathfrak{D}\pi^6}{1.2.3\ldots.7} + \&c.$

129. Ex tabula valorum numerorum α, β, γ, δ, &c. quam ſupra §. 121. exhibuimus, patet eos primum decreſcere tum vero iterum creſcere, & quidem in infinitum. Operae igitur pretium erit inveſtigare, in quanam ratione hi numeri, poſtquam iam vehementer longe fuerint continuati, ulterius progredi pergant. Sit igitur ϕ numerus quicunque huius ſeriei numerorum α, β, γ, δ, &c. longiſſime ab initio remotus, & ſit ψ numerorum ſequens. Quoniam per hos numeros ſummae poteſtatum reciprocarum definiuntur, ſit $2n$ exponens poteſtatis, in cuius ſumma numerus ϕ ingreditur, erit $2n+2$ exponens poteſtatis numero ψ reſpondens, atque numerus n iam erit vehementer magnus. Hinc ex §. 125. habebitur:

$$1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \&c. = \frac{2^{2n-1}\phi}{1.2.3\ldots(2n+1)}\pi^{2n}$$

$$1 + \frac{1}{2^{2n+2}} + \frac{1}{3^{2n+2}} + \frac{1}{4^{2n+2}} + \&c. = \frac{2^{2n+1}\psi}{1.2.3\ldots(2n+3)}\pi^{2n+2}$$

Quod ſi ergo haec per iſtam dividatur, erit:

$$\frac{1 + \frac{1}{2^{2n+2}} + \frac{1}{3^{2n+2}} + \&c.}{1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \&c.} = \frac{4\psi}{(2n+2)(2n+3)}\,\frac{\pi^2}{\phi}$$

Quia vero n eſt numerus vehementer magnus, ob ſeriem utramque proxime $=1$, erit:

$$\frac{\psi}{\phi} = \frac{(2n+2)(2n+3)}{4\pi^2} = \frac{nn}{\pi\pi}$$

Cum igitur n deſignet, quotus ſit numerus ϕ a primo α computatus, ſe habebit hic numerus ϕ ad ſuum ſequentem ψ ut π^2 ad n^2, quae ratio, ſi n fuerit numerus infinitus, veritati penitus ſit conſentanea. Quoniam eſt fere $\pi\pi = 10$,

ſi

ſi ponatur $n = 100$; erit terminus centeſimus circiter millies minor ſuo ſequente. Conſtituunt ergo numeri $\alpha, \beta, \gamma, \delta$, &c. pariter ac Bernoulliani $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, &c. ſeriem maxime divergentem, quae etiam magis increſcat, quam ulla ſeries geometrica terminis creſcentibus procedens.

130. Inventis ergo his valoribus numerorum $\alpha, \beta, \gamma, \delta$, &c. ſeu $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, &c. ſi proponatur ſeries, cuius terminus generalis z fuerit functio quaecunque ipſius indicis x, terminus ſummatorius Sz huius ſeriei ſequenti modo exprimetur, ut ſit:

$$Sz = \int z dx + \tfrac{1}{2} z + \frac{1}{6} \cdot \frac{dz}{1.2\, dx} - \frac{1}{30} \cdot \frac{d^3 z}{1.2.3.4\, dx^3}$$
$$+ \frac{1}{42} \cdot \frac{d^5 z}{1.2.3 \ldots 6 dx^5} - \frac{1}{30} \cdot \frac{d^7 z}{1.2.3 \ldots 8 dx^7}$$
$$+ \frac{5}{66} \cdot \frac{d^9 z}{1.2.3 \ldots 10 dx^9} - \frac{691}{2730} \cdot \frac{d^{11} z}{1.2.3 \ldots 12 dx^{11}}$$
$$+ \frac{7}{6} \cdot \frac{d^{13} z}{1.2.3 \ldots 14 dx^{13}} - \frac{3617}{510} \cdot \frac{d^{15} z}{1.2.3 \ldots 16 dx^{15}}$$
$$+ \frac{43867}{798} \cdot \frac{d^{17} z}{1.2.3 \ldots 18 dx^{17}} - \frac{174611}{330} \cdot \frac{d^{19} z}{1.2.3 \ldots 20 dx^{19}}$$
$$+ \frac{854513}{138} \cdot \frac{d^{21} z}{1.2.3 \ldots 22 dx^{21}} - \frac{236364091}{2730} \cdot \frac{d^{23} z}{1.2.3 \ldots 24 dx^{23}}$$
$$+ \frac{8553103}{6} \cdot \frac{d^{25} z}{1.2.3 \ldots 26 dx^{25}} - \frac{23749461029}{870} \cdot \frac{d^{27} z}{1.2.3 \ldots 28 dx^{27}}$$
$$+ \frac{8615841276005}{14322} \cdot \frac{d^{29} z}{1.2.3 \ldots 30 dx^{19}} - \&c.$$

Si igitur innotuerit integrale $\int z dx$, ſeu quantitas illa cuius differentiale ſit $= z dx$, terminus ſummatorius ope continuae differentiationis invenietur. Perpetuo autem notandum eſt ad hanc expreſſionem ſemper eiuſmodi conſtantem addi oportere, ut ſumma fiat $= 0$, ſi index x ponatur in nihilum abire:

131. Si igitur z fuerit functio rationalis integra ipſius x,

x, quia eius differentialia tandem evaneſcunt, terminus ſummatorius per expreſſionem finitam exprimetur; id quod ſequentibus exemplis illuſtrabimus.

EXEMPLUM I.

Quaeratur terminus ſummatorius huius ſeriei:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & & & x \\ 1 + & 9 + & 25 + & 49 + & 81 + & \cdots & + & (2x-1)^2. \end{array}$$

Quia hic eſt $z = (2x-1)^2 = 4xx - 4x + 1$;

erit $\int z dx = \frac{4}{3}x^3 - 2x^2 + x$,

ex huius enim differentiatione oritur:

$$4xxdx - 4xdx + dx = zdx.$$

Deinde vero per differentiationem erit:

$$\frac{dz}{dx} = 8x - 4$$

$$\frac{ddz}{dx^2} = 8$$

$$\frac{d^3z}{dx^3} = 0 \quad \&c.$$

Hinc erit terminus ſummatorius quaeſitus: $=$

$\frac{4}{3}x^3 - 2x^2 + x + 2xx - 2x + \frac{1}{2} + \frac{2}{3}x - \frac{1}{3} \pm$ Conſt.

qua conſtante tolli debent termini $\frac{1}{2} - \frac{1}{3}$, unde erit

$$S(2x-1)^2 = \frac{4}{3}x^3 - \frac{1}{3}x = \frac{x}{3}(2x-1)(2x+1).$$

Sic erit poſito $x = 4$ ſumma 4 primorum terminorum

$$1 + 9 + 25 + 49 = \frac{4}{3}\cdot 7 \cdot 9 = 84.$$

EXEMPLUM II.

Quaeratur terminus ſummatorius huius ſeriei:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & & & x \\ 1 + & 27 + & 125 + & 343 + & \cdots & & (2x-1)^3 \end{array}$$

Quia eſt $z = (2x-1)^3 = 8x^3 - 12x^2 + 6x - 1$; erit

$\int zdx$

$\int z dx = 2x^4 - 4x^3 + 3x^2 - x$; $\frac{dz}{dx} = 24x^2 - 24x + 6$;

$\frac{ddz}{dx^2} = 48x - 24$; $\frac{d^3z}{dx^3} = 48$; ſequentia evaneſcunt.

Quare erit

$$\begin{aligned} S(2x-1)^3 = {} & 2x^4 - 4x^3 + 3x^2 - x \\ & + 4x^3 - 6x^2 + 3x - \tfrac{1}{2} \\ & + 2x^2 - 2x + \tfrac{1}{2} \\ & - \tfrac{1}{15} \end{aligned}$$

hoc eſt $S(2x-1)^3 = 2x^4 - x^2 = x^2(2xx-1)$. Sic erit poſito $x = 4$ $\quad 1 + 27 + 125 + 343 = 16.31 = 496$.

132. Ex hac inventa generali expreſſione pro termino ſummatorio ſponte ſequitur ille terminus ſummatorius, quem ſuperiori parte pro poteſtatibus numerorum naturalium dedimus, cuiuſque demonſtrationem ibi tradere non licuerat. Quod ſi enim ponamus $z = x^n$, erit utique

$\int z dx = \frac{1}{n+1} x^{n+1}$; differentialia vero ita ſe habebunt:

$$\frac{dz}{dx} = nx^{n-1}$$

$$\frac{ddz}{dx^2} = n(n-1)x^{n-2}$$

$$\frac{d^3z}{dx^3} = n(n-1)(n-2)x^{n-3}$$

$$\frac{d^5z}{dx^5} = n(n-1)(n-2)(n-3)(n-4)x^{n-5}$$

$$\frac{d^7z}{dx^7} = n(n-1) \;\; . \;\; . \;\; . \;\; . \;\; . \;\; (n-6)x^{n-7} \qquad \&c.$$

Ex his ergo deducetur ſequens terminus ſummatorius reſpondens termino generali x^n; ſcilicet

$$Sx^n = \frac{1}{n+1}x^{n+1} + \frac{1}{2}x^n + \frac{1}{6}\cdot\frac{n}{2}x^{n-1}$$

$$-\frac{1}{30}\cdot\frac{n(n-1)(n-2)}{2.\ 3.\ 4}x^{n-3}$$

$$+\frac{1}{42}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)}{2.\ 3.\ 4.\ 5.\ 6}x^{n-5}$$

$$-\frac{1}{30}\cdot\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-6)}{2.\ 3.\ .\ .\ .\ .\ .\ .\ .\ 8}x^{n-7}$$

$$+\frac{5}{66}\cdot\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-8)}{2.\ 3.\ .\ .\ .\ .\ .\ .\ 10}x^{n-9}$$

$$-\frac{691}{2730}\cdot\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-10)}{2.\ 3.\ .\ .\ .\ .\ .\ .\ 12}x^{n-11}$$

$$+\frac{7}{6}\cdot\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-12)}{2.\ 3.\ .\ .\ .\ .\ .\ .\ 14}x^{n-13}$$

$$-\frac{3617}{510}\cdot\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-14)}{2.\ 3.\ .\ .\ .\ .\ .\ .\ 16}x^{n-15}$$

$$+\frac{43867}{798}\cdot\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-16)}{2.\ 3.\ .\ .\ .\ .\ .\ .\ 18}x^{n-17}$$

$$-\frac{174611}{330}\cdot\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-18)}{2.\ 3.\ .\ .\ .\ .\ .\ .\ 20}x^{n-19}$$

$$+\frac{854513}{138}\cdot\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-20)}{2.\ 3.\ .\ .\ .\ .\ .\ .\ 22}x^{n-21}$$

$$-\frac{236364091}{2730}\cdot\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-22)}{2.\ 3.\ .\ .\ .\ .\ .\ .\ 24}x^{n-23}$$

$$+\frac{8553103}{6}\cdot\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-24)}{2.\ 3.\ .\ .\ .\ .\ .\ .\ 26}x^{n-25}$$

$$-\frac{23749461029}{870}\cdot\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-26)}{2.\ 3.\ .\ .\ .\ .\ .\ .\ 28}x^{n-27}$$

$$+\frac{8615841276005}{14322}\cdot\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-28)}{2.\ 3.\ .\ .\ .\ .\ .\ .\ 30}x^{n-29}\ \&c.$$

quae

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quae expreſſio non differt ab ea, quam ſupra dedimus, niſi quod hic numeros Bernoullianos $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. introduximus, cum ſupra uſi eſſemus numeris α, β, γ, δ, &c. interim tamen conſenſus ſponte elucet. Hinc ergo terminos ſummatorios omnium poteſtatum uſque ad poteſtatem trigeſimam incluſive exhibere licuit; quae inveſtigatio, ſi alia via fuiſſet ſuſcepta, ſine longiſſimis & taedioſiſſimis calculis abſolvi non potuiſſet.

133. Iam ſupra §. 59. ſimilem fere expreſſionem pro termino ſummatorio dedimus ex termino generali definiendo. Ea enim pariter ſecundum differentialia termini generalis procedebat; ab iſta autem in hoc potiſſimum erat diverſa, quod illa non integrale $\int z dx$ requirebat, ſingula vero termini generalis differentialia per certas ipſius x functiones habebat multiplicata. Eandem igitur expreſſionem ſequenti modo ad naturam ſerierum magis accommodato denuo eliciamus, ex quo ſimul lex clarius patebit, ſecundum quam coefficientes illi differentialium progrediuntur. Sit igitur ſeriei terminus generalis z, functio quaecunque ipſius indicis x, terminus vero ſummatorius quaeſitus ſit s: qui quoniam uti vidimus eiuſmodi erit functio ipſius x, ut evaneſcat poſito $x = 0$, erit per ea, quae ſupra de natura huiuſmodi functionum demonſtravimus:

$$s - \frac{xds}{1dx} + \frac{x^2 dds}{1.2dx^2} - \frac{x^3 d^3 s}{1.2.3dx^3} + \frac{x^4 d^4 s}{1.2.3.4dx^4} - \&c. = 0.$$

134. Quia s denotat ſummam omnium terminorum ſeriei a primo uſque ad ultimum z, perſpicuum eſt ſi in s loco x ponatur $x - 1$, tum priorem ſummam ultimo termino z mulctari: erit ſcilicet

$$s - z = s - \frac{ds}{dx} + \frac{dds}{2dx^2} - \frac{d^3 s}{6dx^3} + \frac{d^4 s}{24dx^4} - \&c.$$

ideoque $$z = \frac{ds}{dx} - \frac{dds}{2dx^2} + \frac{d^3 s}{6dx^3} - \frac{d^4 s}{24dx^4} + \&c.$$

quae aequatio modum ſuppeditat ex dato termino ſummatorio s definiendi terminum generalem, quod quidem per ſe eſt facillimum. Ex idonea autem combinatione huius aequationis cum ea, quam §. praeced. invenimus, valor ipſius s per x & z determinari poterit. Ponamus in hunc finem eſſe:

$$s - Az + \frac{Bdz}{dx} - \frac{Cddz}{dx^2} + \frac{Dd^3z}{dx^3} - \frac{Ed^4z}{dx^4} + \&c. = 0.$$

ubi A, B, C, D, &c. denotent coefficientes neceſſario ſive conſtantes ſive variabiles: nam cum ſit

$$z = \frac{ds}{dx} - \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} - \frac{d^4s}{24dx^4} + \frac{d^5s}{120dx^5} - \&c.$$

ſi hinc valores pro z, $\frac{dz}{dx}$, $\frac{ddz}{dx^2}$, $\frac{d^3z}{dx^3}$ &c. in ſuperiori aequatione ſubſtituantur, prodibit:

$$s = s$$

$$-Az = -\frac{Ads}{dx} + \frac{Adds}{2dx^2} - \frac{Ad^3s}{6dx^3} + \frac{Ad^4s}{24dx^4} - \frac{Ad^5s}{120dx^5} + \&c.$$

$$+\frac{Bdz}{dx} = +\frac{Bdds}{dx^2} - \frac{Bd^3s}{2dx^3} + \frac{Bd^4s}{6dx^4} - \frac{Bd^5s}{24dx^5} + \&c.$$

$$-\frac{Cddz}{dx^2} = -\frac{Cd^3s}{dx^3} + \frac{Cd^4s}{2dx^4} - \frac{Cd^5s}{6dx^5} + \&c.$$

$$+\frac{Dd^3z}{dx^3} = +\frac{Dd^4s}{dx^4} - \frac{Dd^5s}{2dx^5} + \&c.$$

$$-\frac{Ed^4z}{dx^4} = -\frac{Ed^5s}{dx^5} + \&c.$$

&c.

quae igitur ſeries iunctim ſumtae aequales erunt nihilo.

135. Cum ergo ante invenimus eſſe:

$$0 = s - \frac{xds}{dx} + \frac{x^2dds}{2dx^2} - \frac{x^3d^3s}{6dx^3} + \frac{x^4d^4s}{24dx^4} - \frac{x^5d^5s}{120dx^5} + \&c.$$

ſi

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ſi ſuperior aequatio huic aequalis ſtatuatur, prodibunt ſequentes litterarum A, B, C, D, &c. denominationes:

$$A = x$$

$$B = \frac{x^2}{2} - \frac{A}{2}$$

$$C = \frac{x^3}{6} - \frac{B}{2} - \frac{A}{6}$$

$$D = \frac{x^4}{24} - \frac{C}{2} - \frac{B}{6} - \frac{A}{24}$$

$$E = \frac{x^5}{120} - \frac{D}{2} - \frac{C}{6} - \frac{B}{24} - \frac{A}{120} \qquad \&c.$$

His igitur litterarum A, B, C, D, &c. valoribus inventis, ex termino generali z terminus ſummatorius $s = Sz$ ita determinabitur, ut ſit:

$$Sz = Az - \frac{B\,dz}{dx} + \frac{C\,ddz}{dx^2} - \frac{D\,d^3z}{dx^3} + \frac{E\,d^4z}{dx^4} - \frac{F\,d^5z}{dx^5} + \&c.$$

136. Cum autem fiat:

$$A = x$$

$$B = \tfrac{1}{2}x^2 - \tfrac{1}{2}x$$

$$C = \tfrac{1}{6}x^3 - \tfrac{1}{4}x^2 + \tfrac{1}{12}x$$

$$D = \tfrac{1}{24}x^4 - \tfrac{1}{12}x^3 + \tfrac{1}{24}xx \qquad \&c.$$

patet hos coefficientes eſſe eoſdem, quos ſupra §. 59. habuimus, unde iſta termini ſummatorii expreſſio eadem eſt, quam ibi invenimus; eritque propterea:

$$A = Sx^0 = S1$$

$$B = \tfrac{1}{1}Sx^1 - \tfrac{1}{1}x$$

$$C = \tfrac{1}{2}Sx^2 - \tfrac{1}{2}x^2$$

$$D = \tfrac{1}{6}Sx^3 - \tfrac{1}{6}x^3$$

$$E = \tfrac{1}{24}Sx^4 - \tfrac{1}{24}x^4 \qquad \&c.$$

Hinc ergo erit:

$$Sz = xz - \frac{dz}{dx}Sx + \frac{ddz}{2dx^2}Sx^2 - \frac{d^3z}{6dx^3}Sx^3 + \frac{d^4z}{24dx^4}Sx^4 - \&c.$$

$$+\frac{xdz}{dx}-\frac{x^2ddz}{2dx^2}+\frac{x^3d^3z}{6dx^3}-\frac{x^4d^4z}{24dx^4}+\&c.$$

Quodſi autem in termino generali z ponatur $x=0$, prodibit terminus indici $=0$ reſpondens; qui ſi ponatur $=a$, erit: $a=z-\frac{xdz}{dx}+\frac{x^2ddz}{2dx^2}-\frac{x^3d^3z}{6dx^3}+\&c.$ ideoque

$$\frac{xdz}{dx}-\frac{x^2ddz}{2dx^2}+\frac{x^3d^3z}{6dx^3}-\frac{x^4d^4z}{24dx^4}+\&c.=z-a,$$

quo valore ſubſtituto habebitur:

$$Sz=(x+1)z-a-\frac{dz}{dx}Sx+\frac{ddz}{2dx^2}Sx^2-\frac{d^3z}{6dx^3}Sx^3+\frac{d^4z}{24dx^4}Sx^4-\&c.$$

Cognitis ergo ſummis poteſtatum, hinc pro quovis termino generali ei conveniens terminus ſummatorius exhiberi poteſt.

137. Quoniam ergo geminam invenimus expreſſionem termini ſummatorii Sz pro termino generali z, earumque altera formulam integralem $\int zdx$ continet, ſi iſtae duae expreſſiones ſibi aequales ponantur, obtinebitur valor ipſius $\int zdx$ per ſeriem expreſſus. Cum enim ſit:

$$\int zdx+\tfrac{1}{2}z+\frac{\mathfrak{A}dz}{1.2dx}-\frac{\mathfrak{B}d^3z}{1.2.3.4dx^3}+\frac{\mathfrak{C}d^5z}{1.2\ldots6dx^5}-\&c.$$

$$=(x+1)z-a-\frac{dz}{1dx}Sx+\frac{ddz}{1.2dx^2}Sx^2-\frac{d^3z}{1.2.3dx^3}Sx^3+\&c.$$

erit:

$$\int zdx=(x+\tfrac{1}{2})z-a-\frac{dz}{dx}(Sx+\tfrac{1}{2}\mathfrak{A})+\frac{ddz}{2dx^2}Sx^2-\frac{d^3z}{6dx^3}(Sx^3-\tfrac{1}{4}\mathfrak{B})$$

$$+\frac{d^4z}{24dx^4}Sx^4-\frac{d^5z}{120dx^5}(Sx^5+\tfrac{1}{6}\mathfrak{C})+\frac{d^6z}{720dx^6}Sx^6$$

$$-\frac{d^7z}{5040dx^7}(Sx^7-\tfrac{1}{8}\mathfrak{D})+\&c.$$

ubi $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. denotant numeros Bernoullianos ſupra §. 122. exhibitos.

Sit

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Sit v. gr. $z = xx$, fiet $a = 0$; $\frac{dz}{dx} = 2x$; & $\frac{ddz}{2dx^2} = 1$, hinc erit:
$\int xxdx = (x + \frac{1}{2})xx - 2x(\frac{1}{2}xx + \frac{1}{2}x + \frac{1}{12}) + 1(\frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x)$ ſeu $\int xxdx = \frac{1}{3}x^3$; dat autem $\frac{1}{3}x^3$ differentiatum utique $xxdx$.

138. Nova ergo hinc patet via ad terminos ſummatorios ſerierum poteſtatum inveniendos; quoniam enim ex coefficientibus ante aſſumtis A, B, C, D, &c. hi termini ſummatorii facillime formantur, horum autem coefficientium quilibet ex praecedentibus conflatur; ſi in formulis §. 135. datis loco iſtarum litterarum valores in §. 136. traditi ſubſtituantur, erit:

$$Sx^1 - x = \frac{1}{2}xx - \frac{1}{2}x$$

$$Sx^2 - x^2 = \frac{1}{3}x^3 - \frac{1}{3}x - \frac{2}{2}(Sx - x)$$

$$Sx^3 - x^3 = \frac{1}{4}x^4 - \frac{1}{4}x - \frac{3}{2}(Sx^2 - x^2) - \frac{3\cdot 2}{2\cdot 3}(Sx^1 - x^1)$$

$$Sx^4 - x^4 = \frac{1}{5}x^5 - \frac{1}{5}x - \frac{4}{2}(Sx^3 - x^3) - \frac{4\cdot 3}{2\cdot 3}(Sx^2 - x^2) - \frac{4\cdot 3\cdot 2}{2\cdot 3\cdot 4}(Sx - x)$$

&c.

Hinc ergo ſummae poteſtatum ſuperiorum ex ſummis inferiorum formari poterunt.

139. Quod ſi vero legem, qua coefficientes A, B, C, D, &c. ſupra §. 135. progredi inventi ſunt, attentius intueamur, eos ſeriem recurrentem conſtituere deprehendemus. Si enim evolvamus hanc fractionem:

$$\frac{x + \frac{1}{2}xxu + \frac{1}{6}x^3u^2 + \frac{1}{24}x^4u^3 + \frac{1}{120}x^5u^4 + \&c.}{1 + \frac{1}{2}u + \frac{1}{6}u^2 + \frac{1}{24}u^3 + \frac{1}{120}u^4 + \&c.}$$

ſecundum poteſtates ipſius u, hancque ſeriem reſultare ſumamus.

$$A + Bu + Cu^2 + Du^3 + Eu^4 + \&c.$$

erit uti ante invenimus $A = x$; $B = \frac{1}{2}xx - \frac{1}{2}A$; &c. ſicque inventa hac ſerie, obtinebuntur termini ſummatorii ſerierum poteſtatum. Illa autem fractio, ex cuius evolutione iſta ſeries naſcitur, tranſit in hac formam: $\frac{e^{xu} - 1}{e^u - 1}$, quae ſi x fuerit

rit numerus integer affirmativus, abit in
$1+e^{u}+e^{2u}+e^{3u}+\ldots+e^{(x-1)u}$; cum ergo ſit:

$$1=1$$

$$e^{u}=1+\frac{u}{1}+\frac{u^2}{1.2}+\frac{u^3}{1.2.3}+\frac{u^4}{1.2.3.4}+\&c.$$

$$e^{2u}=1+\frac{2u}{1}+\frac{4u^2}{1.2}+\frac{8u^3}{1.2.3}+\frac{16u^4}{1.2.3.4}+\&c.$$

$$e^{3u}=1+\frac{3u}{1}+\frac{9u^2}{1.2}+\frac{27u^3}{1.2.3}+\frac{81u^4}{1.2.3.4}+\&c.$$

$$e^{(x-1)u}=1+\frac{(x-1)u}{1}+\frac{(x-1)^2u^2}{1.2}+\frac{(x-1)^3u^3}{1.2.3}+\frac{(x-1)^4u^4}{1.2.3.4}+\&c.$$

ideoque erit

$$A=x$$
$$B=S(x-1)=Sx-x$$
$$C=\tfrac{1}{2}S(x-1)^2=\tfrac{1}{2}Sx^2-\tfrac{1}{2}x^2$$
$$D=\tfrac{1}{6}S(x-1)^3=\tfrac{1}{6}Sx^3-\tfrac{1}{6}x^3$$
&c.

Unde nexus horum coefficientium cum ſummis poteſtatum, ante iam obſervatus, penitus confirmatur ac demonſtratur.

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